Questions:

What are some important Laplace transforms?

What is a transfer function?

What are poles?

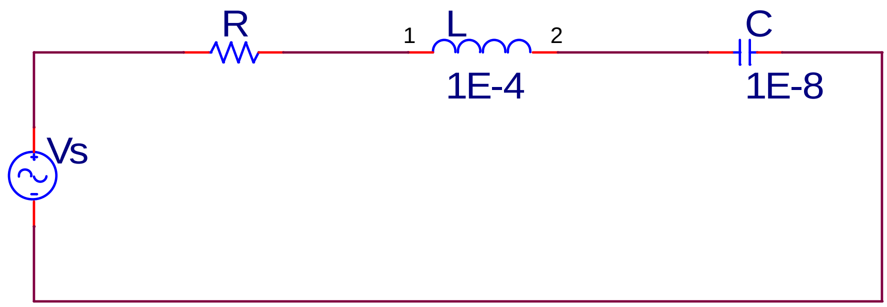
What are zeros?

Why is the left half plane significant?

Review RLC series circuit: 2nd order differential equations

1)

RLC series circuits



In the above circuit, the voltage source is $Vs = \begin{cases} 5V & t < 0 \\ 10V & 0 < t \end{cases}$

$L_1 := 1 \cdot 10^{-4} H$ $\qquad$ $C_1 := 1 \cdot 10^{-8} F$

a. For what range of resistor values is the circuit overdamped?

The circuit is overdamped when α>ω0, giving

$\alpha = \frac{R}{2L}$ $\qquad$ $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

$$R > \frac{2L}{\sqrt{LC}} \qquad \frac{2 \cdot 1 \cdot 10^{-4}}{\sqrt{1 \cdot 10^{-4} \cdot 1 \cdot 10^{-8}}} = 200$$

$R > 200$

b. For what resistor value is the circuit critically damped?

The circuit is critically damped when α=ω0

$R = \frac{2L}{\sqrt{LC}}$ $\qquad$ $R = 200\Omega$

c. For what range of resistor values is the circuit underdamped?

The circuit is underdamped when $\alpha<\omega 0$

$$R < 200\Omega$$

Determine the voltage across the capacitor as a function of time, Vc(t), when

d. R=2000Ω

$R_d := 2000\Omega$ (circuit will be overdamped)

$$\alpha := \frac{R_d}{2L_1}$$

$$\alpha = 1 \times 10^7 \frac{1}{s}$$

$$\omega_0 := \frac{1}{\sqrt{L_1 \cdot C_1}}$$

$$\omega_0 = 1 \times 10^6 \frac{1}{s}$$

$$s_{1d} := -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_{1d} = -5.013 \times 10^4 \frac{1}{s}$$

$$s_{2d} := -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$s_{2d} = -1.995 \times 10^7 \frac{1}{s}$$

Overdamped takes the form of two decaying exponentials

$$V_c(t) = A_1 \cdot \exp\left(-5.013 \cdot 10^4 t\right) + A_2 \cdot \exp\left(-1.99 \cdot 10^7 t\right) + A_3$$

The t goes to ∞ DC steady state solution give the A3 term A3=10

The inital conditions give

$$V_C\left(0^+\right) = A_1 \cdot e^0 + A_2 \cdot e^0 + 10 = 5$$

$$dV_C(0^+) = \frac{I_C(0^+)}{C} = \frac{I_L(0^+)}{C} = 0 = -5.01\cdot 10^4\cdot A_1 - 1.99\cdot 10^7\cdot A_2$$

this equation is the derivative of the one above it, the left side comes from the definition of the capacitor and the series condition of the RLC

$$M_1 := \begin{pmatrix} 1 & 1 \\ -5.01\cdot 10^4 & -1.99\cdot 10^7 \end{pmatrix}$$

$$v_{CN} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$C_d := \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

$$X_1 := M_1^{-1}\cdot C_d$$

$$X_1 = \begin{pmatrix} -5.013 \\ 0.013 \end{pmatrix}$$

$$A_{1d} := -5.013$$

$$A_{2d} := 0.013$$

$$A_3 := 10$$

$$V_c(t) = -5.013\cdot \exp\left(-5.013\cdot 10^4 t\right) + 0.013\cdot \exp\left(1.99\cdot 10^7 t\right) + 10$$

Determine the voltage across the capacitor as a function of time, Vc(t), when

e. R=200Ω

$R_e := 200\Omega$ we know from above that this is critically damped

$$\alpha := \frac{R_e}{2L_1}$$

$$\alpha = 1\times 10^6 \frac{1}{s}$$

$$\omega_0 := \frac{1}{\sqrt{L_1 \cdot C_1}}$$

$$\omega_0 = 1\times 10^6 \frac{1}{s}$$

$$V_c(t) = A_1 \cdot \exp\left(-1\cdot 10^6 t\right) + A_2 \cdot t \cdot \exp\left(-1\cdot 10^6 t\right) + A_3$$

The t goes to ∞ DC steady state solution give the A3 term, A3 =10

The initial conditions give

$$V_C\left(0^+\right) = A_{1e}{}^0 + A_2 \cdot 0 \cdot e^0 + 10 = 5$$

$$A_1 + 10 = 5$$

$$A_1 := -5$$

$$dV_C\left(0^+\right) = \frac{I_C\left(0^+\right)}{C} = \frac{I_L\left(0^+\right)}{C} = 0 = -1\cdot 10^6 \cdot A_1 + A_2$$

$$0 = -5\cdot\left(-1\cdot 10^6\right) + A_2$$

$$A_2 = -5\cdot 10^6$$

$$V_c(t) = -5\cdot \exp\left(-1\cdot 10^6 t\right) - 5\cdot 10^6 \cdot \exp\left(-1\cdot 10^6 t\right) + 10$$

Determine the voltage across the capacitor as a function of time, Vc(t), when

f. R=20Ω

$R_f := 20\Omega$ we know from above that this is underdamped

$$\alpha := \frac{R_f}{2L_1}$$

$$\alpha = 1\times 10^5\frac{1}{s}$$

$$\omega_0 := \frac{1}{\sqrt{L_1\cdot C_1}}$$

$$\omega_0 = 1\times 10^6\frac{1}{s}$$

$$\beta := \sqrt{\omega_0{}^2 - \alpha^2}$$

$$\beta = 9.95\times 10^5\frac{1}{s}$$

$$V_c(t) = \exp^{-1\cdot 10^5 t}\cdot\left(\left(A_1\cdot\cos\left(9.95\cdot 10^5 t\right) + A_2\cdot\sin\left(9.95\cdot 10^5 t\right)\right)\right) + A_3$$

The t goes to ∞ DC steady state solution give the A3 term A3 = 10

The initial conditions give

$$V_C\left(0^+\right) = A_1 + 10 = 5 \qquad \cos(0) = 1 \qquad \sin(0) = 0$$

$$A_1 = -5$$

$$\frac{dV_C\left(0^+\right)}{dt} = \frac{I_C\left(0^+\right)}{C} = \frac{I_L\left(0^+\right)}{C} = 0 = -1\cdot 10^5\cdot A_1 + 9.95\cdot 10^5 A_2$$

$$-5\cdot\left(-1\cdot 10^5\right)\cdot A_1 + 9.95\cdot 10^5 A_2 = 0$$

$$A_2 = 0.503$$

$$V_c(t) = \exp^{-1\cdot 10^5 t}\cdot\left(\left(-5\cdot\cos\left(9.95\cdot 10^5 t\right) + 0.503\cdot\sin\left(9.95\cdot 10^5 t\right)\right)\right) + 10$$

2) Find the Laplace transform of the following function

$$f(t) = (5\cdot\exp(-5t) - 10t\cdot\exp(-5t) + 10)u(t)$$

multiply it out

$$5\cdot\exp(-5t)\cdot u(t) - 10t\cdot\exp(-5t)\cdot u(t) + 10\cdot u(t)$$

$\exp(-\alpha t)\,u(t)$ mean +5

$$\frac{5}{s+5} + \frac{-10}{(s+5)^2} + \frac{10}{s}$$

3). $$F(s) = \frac{2s+1}{s^3 + 6s^2 + 8s}$$

What are the poles and zeros

Zeros $2s + 1 = 0$ $s = \frac{-1}{2}$ $z = -0.5$

Poles $s\left(s^2 + 6s + 8\right)$ $p_1 = 0$

$s\ \ (s+4)(s+2)$ $p_2 = -4$

$p_3 = -2$

Write F(s) as

$$\frac{2(s+0.5)}{s\cdot(s+2)\cdot(s+4)}$$

Draw the pole zero diagram

poles are on the real axis and negative
overdamped response
No imaginary term

pole at zero implies a DC term

Using partial fraction expansion find f(t) including values for all A coefficients

$$\frac{2(s+0.5)}{s\cdot(s+2)\cdot(s+4)} = \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+4}$$

$$L^{-1}\cdot\{F(s)\} = L^{-1}\cdot\left(\frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+4}\right)$$ break into parts

$$L^{-1}\left(\frac{A_1}{s}\right) + L^{-1}\left(\frac{A_2}{s+2}\right) + L^{-1}\cdot\left(\frac{A_3}{s+4}\right) = A_1\cdot u(t) + A_2\cdot e^{-2t}u(t) + A_3\cdot e^{-4t}u(t)$$

$$= \left(A_1 + A_2\cdot e^{-2t} + A_3\cdot e^{-4t}\right)\cdot u(t)$$

Use the cover up rule

Find A1, multiply both sides by s (denominator of A1 term), evaluate at the pole, s=0

A_1

$$\frac{2(s+0.5)\cdot s}{s\cdot(s+2)\cdot(s+4)} = A_1 + \frac{A_2\cdot s}{s+2} + \frac{A_3\cdot s}{s+4}$$ at s =0

$$\frac{1}{8} = A_1 \qquad A_1 = 0.125$$

Find A2, multiply by s+2, evaluate at -2

A2

$$\frac{2(s+0.5)(s+2)}{s\cdot(s+2)\cdot(s+4)} = A_1(s+2) + \frac{A_2\cdot(s+2)}{s+2} + \frac{A_3\cdot(s+2)}{s+4}$$ at s=-2

$$A_2 = 0.75$$

Find A3, multiply by s+4, evaluate at s=-4

$$A_3 = -0.875$$

$$F(s) = \frac{0.125}{s} + \frac{0.75}{s+2} - \frac{0.875}{s+4}$$

$$f(t) = \left(0.125 + 0.75e^{-2t} - 0.875 \cdot e^{-4t}\right) \cdot u(t)$$

poles are real and negative, exponentially decaying terms

What happens with a pole at +4

$$F(s) = \frac{A_1}{s-4}$$

$L^{-1} = A_1 \cdot e^{4t}$ not decaying!!! Not stable

For a stable response poles must be in the left half of the pole zero diagram

Real component of a pole must be negative

3) $$F(s) = \frac{s+1}{(s+4)\cdot(s+3)\cdot(s+2)}$$

Zeros -1

Poles $-2, -3, -4$

$$\frac{s+1}{(s+4)\cdot(s+3)\cdot(s+2)} = \frac{A_1}{s+4} + \frac{A_2}{s+3} + \frac{A_3}{s+2}$$

Find A1 using the cover up rule

$\frac{s+1}{(s+3)\cdot(s+2)} = A_1$ at s = -4 $A_1 = \frac{-4+1}{(-4+3)\cdot(-4+2)} = -1.5$

Find A2 using the cover up rule

$\frac{s+1}{(s+4)\cdot(s+2)} = A_2$ at s = -3 $A_2 = \frac{-3+1}{(-3+4)\cdot(-3+2)} = 2$

Find A3 using the cover up rule

$$\frac{s+1}{(s+4)(s+3)} = A_3 \quad \text{at s=-2} \qquad A_3 = \frac{-2+1}{(-2+4)(2+3)} = 0 \cdot -0.5$$

$$f(t) = (-1.5 \cdot \exp(-4t) + 2 \cdot \exp(-3t) - 0.5 \exp(-2t)) \cdot u(t)$$

Pole zero diagram