

# ELECTRIC CIRCUITS

## ECSE-2010

Lecture 10.1

 Rensselaer

### LECTURE 10.1 AGENDA

- Solving process
- Sinusoidal response

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### DYNAMIC CIRCUITS

$$y(t) = y_H + y_P$$

Homogeneous Response + Particular Response

$$y(t) = y_N + y_F$$

Natural Response + Forced Response

$$y_N = y_H; \quad y_F = y_P$$

$$y(t) = y_{ZI} + y_{ZS}$$

Zero-Input Response + Zero-State Response

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### SOLVING PROCESS

Step 1: Draw Circuit at  $t = t_{t_0^-} = 0^-$

Find  $i_L(0^-), v_C(0^-)$

Step 2: Draw Circuit at  $t_{t_0^+} = 0^+$ :

Find  $i_{t_0}, v_{t_0}$

Step 3: Draw Circuit at  $t \rightarrow \infty$

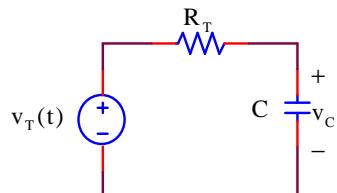
Find  $i_{ss}, v_{ss}$

Step 4: Draw Dead Network

Find  $R_{eq}, \tau_1$

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### SINUSOIDAL RESPONSE



$$v_T(t) = V_A \cos \omega t u(t) \quad t \geq 0$$

Sinusoidal Input That Turns On at  $t = 0$

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### SINUSOIDAL RESPONSE

$$v_C(t) = v_{CN}(t) + v_{CF}(t)$$

$$v_{CN}(t) = K e^{-t/\tau} \quad \tau = R_T C$$

$v_{CF}(t)$  Must Look Like Input

$$v_{CF}(t) = V_F \cos(\omega t + \phi) = a \cos \omega t + b \sin \omega t$$

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## SINUSOIDAL RESPONSE

$$v_{CF}(t) = a \cos \omega t + b \sin \omega t$$

$$\frac{dv_{CF}}{dt} + \frac{1}{R_T C} v_{CF} = \frac{V_T}{R_T C} = \frac{V_A}{R_T C} \cos \omega t$$

$\downarrow$        $\downarrow$        $\downarrow$

$$(-\omega a \sin \omega t + \omega b \cos \omega t) + \frac{1}{\tau} (a \cos \omega t + b \sin \omega t) = \frac{V_A}{\tau} \cos \omega t$$



## SINUSOIDAL RESPONSE

$$(-\omega a \sin \omega t + \omega b \cos \omega t) + \frac{1}{\tau} (a \cos \omega t + b \sin \omega t) = \frac{V_A}{\tau} \cos \omega t$$

Combine  $\cos \omega t$  and  $\sin \omega t$  terms

$$\left( \omega b + \frac{a}{\tau} - \frac{V_A}{\tau} \right) \cos \omega t + \left( -\omega a + \frac{b}{\tau} \right) \sin \omega t = 0$$

$\downarrow$        $\downarrow$

2 Equations  
Solve for a and b



## SINUSOIDAL RESPONSE

$$\left( \omega b + \frac{a}{\tau} - \frac{V_A}{\tau} \right) = 0 \quad \left( -\omega a + \frac{b}{\tau} \right) = 0$$

$$a = \frac{V_A}{1 + \omega^2 \tau^2} \quad b = \frac{\omega \tau V_A}{1 + \omega^2 \tau^2}$$

$$v_C(t) = v_{CN}(t) + v_{CF}(t)$$

$\downarrow$        $\downarrow$

$$v_C(t) = K e^{-t/\tau} + \frac{V_A}{1 + \omega^2 \tau^2} (\cos \omega t + \omega \tau \sin \omega t)$$

Exponential + Sinusoid



## SINUSOIDAL RESPONSE

$$v_C(t) = K e^{-t/\tau} + \frac{V_A}{1 + \omega^2 \tau^2} (\cos \omega t + \omega \tau \sin \omega t)$$

Need to find K; Use Initial Value

$$v_{C0} = V_0 = K + \frac{V_A}{1 + \omega^2 \tau^2}$$

$$K = V_0 - \frac{V_A}{1 + \omega^2 \tau^2}$$



## SINUSOIDAL RESPONSE

$$v_C(t) = \left( V_0 - \frac{V_A}{1 + \omega^2 \tau^2} \right) e^{-t/\tau} + \frac{V_A}{1 + \omega^2 \tau^2} (\cos \omega t + \omega \tau \sin \omega t)$$

Natural Response

Forced Response

(with  $v_{CF}$  in Standard Form)

$$v_C(t) = \left( V_0 - \frac{V_A}{1 + \omega^2 \tau^2} \right) e^{-t/\tau} + \frac{V_A}{\sqrt{1 + \omega^2 \tau^2}} \cos(\omega t + \theta)$$

$$\theta = \tan^{-1}(-\omega \tau)$$



## SINUSOIDAL RESPONSE

Natural Response      Forced Response

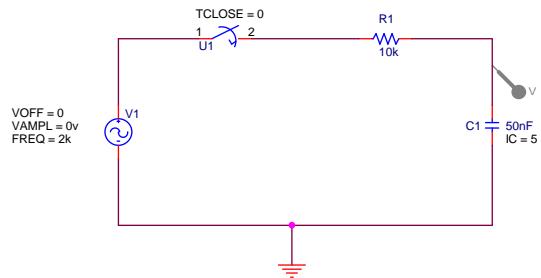
$$v_C(t) = \left( V_0 - \frac{V_A}{1 + \omega^2 \tau^2} \right) e^{-t/\tau} + \frac{V_A}{\sqrt{1 + \omega^2 \tau^2}} \cos(\omega t + \theta)$$

Natural Response  $\rightarrow 0$  in  $\approx 5\tau$

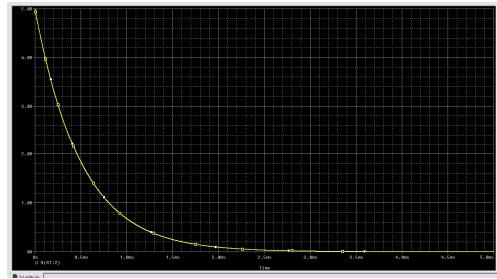
Forced Response  $\rightarrow$  Sinusoid at same  $f$  as Input  
Different Amplitude and Phase  
Forced Response  $\rightarrow$  AC Steady State Response



## SINUSOIDAL RESPONSE



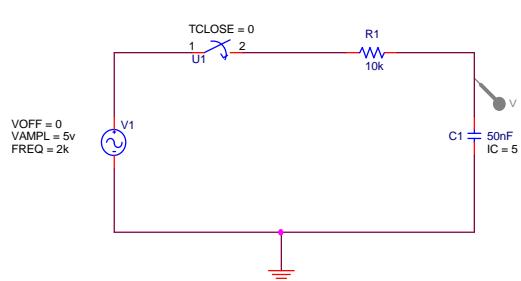
## SINUSOIDAL RESPONSE



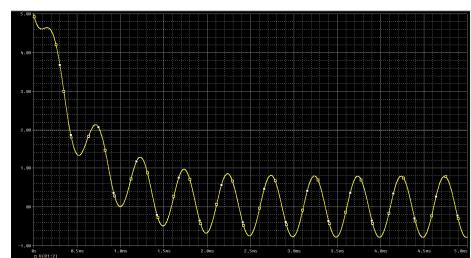
Natural Response



## SINUSOIDAL RESPONSE



## SINUSOIDAL RESPONSE



Natural + Forced Response



# ELECTRIC CIRCUITS

## ECSE-2010

Lecture 10.2



## LECTURE 10.2 AGENDA

- Series RLC Circuits

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## SERIES RLC CIRCUITS

2<sup>nd</sup> Order Circuits

Series RLC Circuits

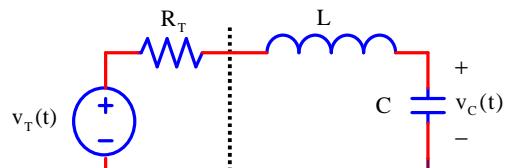


$x(t)$  can be a  
Current or Voltage

$y(t)$  can be a  
Current or Voltage



## SERIES RLC CIRCUITS

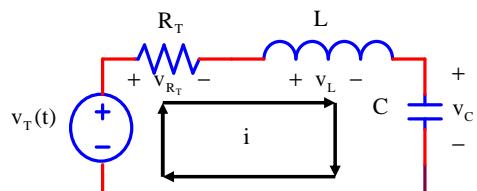


Thevenin  
Equivalent  
Circuit

Find the Differential Equation  
Relating  $v_C(t)$  to  $v_T(t)$



## SERIES RLC CIRCUITS



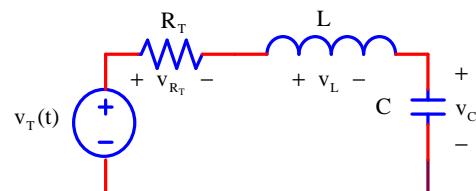
$$\text{KVL: } v_C + v_L + v_{R_T} = v_T$$

$$v_L = L \frac{di}{dt} = LC \frac{d^2v_C}{dt^2}$$

$$v_{R_T} = iR_T = R_T C \frac{dv_C}{dt}$$



## SERIES RLC CIRCUITS



$$\text{KVL: } v_C + v_L + v_{R_T} = v_T$$

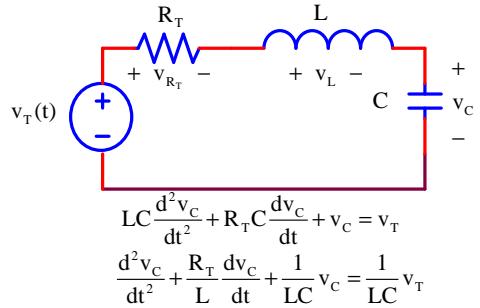
$$LC \frac{d^2v_C}{dt^2} + R_T C \frac{dv_C}{dt} + v_C = v_T$$

2<sup>nd</sup> Order Differential Equation

Constant Coefficients



## SERIES RLC CIRCUITS



## SERIES RLC CIRCUITS

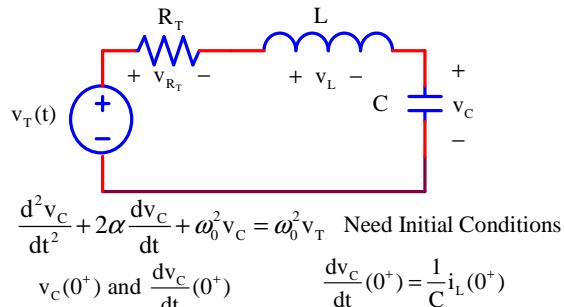
$$\frac{d^2v_C}{dt^2} + \frac{R_T}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} v_T$$

$$\left[ \frac{1}{LC} \right] = \frac{1}{(\text{seconds})^2} = \omega_0^2 \quad \left[ \frac{R_T}{L} \right] = \frac{1}{\text{seconds}} = 2\alpha$$

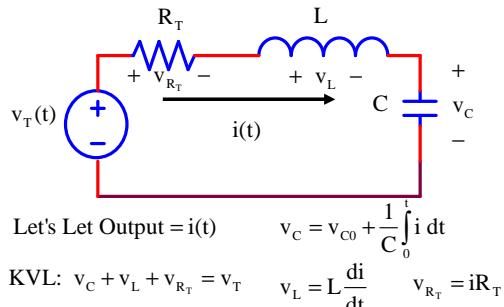
$$\frac{d^2v_C}{dt^2} + 2\alpha \frac{dv_C}{dt} + \omega_0^2 v_C = \omega_0^2 v_T$$



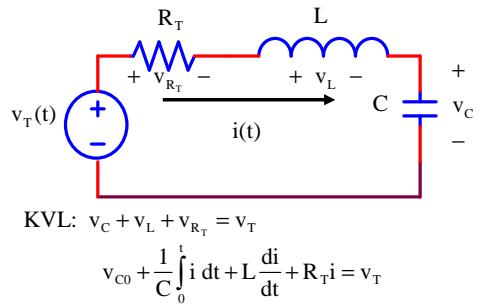
## SERIES RLC CIRCUITS



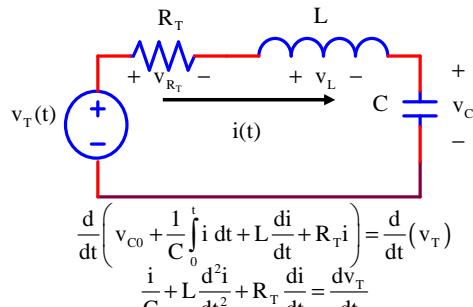
## SERIES RLC CIRCUITS



## SERIES RLC CIRCUITS



## SERIES RLC CIRCUITS



## SERIES RLC CIRCUITS

$$\frac{i}{C} + L \frac{d^2 i}{dt^2} + R_T \frac{di}{dt} = \frac{dv_T}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{R_T}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{LC} \frac{dv_T}{dt}$$

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = \omega_0^2 \frac{dv_T}{dt}$$

LHS is the Same Form as Equation for  $v_C$

Any Current or Voltage in Series RLC Circuit  
Has the Same form for the LHS for its Differential Equation



## SERIES RLC CIRCUITS

$$\frac{d^2 v_{CN}}{dt^2} + 2\alpha \frac{dv_{CN}}{dt} + \omega_0^2 v_{CN} = 0$$

Assume  $v_{CN}(t) = K e^{st}$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Characteristic Equation

$$\text{Roots are } s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$v_{CN}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$



## SERIES RLC CIRCUITS

Case 1:  $\alpha^2 > \omega_0^2$ : 2 Real, Unequal Roots

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$v_{CN} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

2 Decaying Exponentials

Circuit is Overdamped



## SERIES RLC CIRCUITS

$$\frac{d^2 v_C}{dt^2} + 2\alpha \frac{dv_C}{dt} + \omega_0^2 v_C = \omega_0^2 v_T$$

$v_C(t)$  = Natural Response + Forced Response

Let's Look at the Natural Response



## SERIES RLC CIRCUITS

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\text{Roots are } s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

3 Possible Cases:

Case 1:  $\alpha^2 > \omega_0^2$ : 2 Real, Unequal Roots

Case 2:  $\alpha^2 = \omega_0^2$ : 2 Real, Equal Roots

Case 3:  $\alpha^2 < \omega_0^2$ : 2 Complex Conjugate Roots



## SERIES RLC CIRCUITS

Case 1:  $\alpha^2 > \omega_0^2$ : 2 Real, Unequal Roots

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$v_{CN} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

2 Decaying Exponentials

Circuit is Overdamped



## SERIES RLC CIRCUITS

Case 2:  $\alpha^2 = \omega_0^2$ : 2 Real, Equal Roots

$$s_1 = -\alpha \quad \alpha = \frac{R_T}{2L} \quad \omega_0^2 = \frac{1}{LC}$$

$$s_2 = -\alpha$$

$$v_{CN} = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t}$$

Decaying Exponential + Exponentially Damped Ramp

Circuit is Critically Damped



## SERIES RLC CIRCUITS

Case 3:  $\alpha^2 < \omega_0^2$ : 2 Complex Conjugate Roots

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} = -\alpha + j\beta$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - j\sqrt{\omega_0^2 - \alpha^2} = -\alpha - j\beta$$

$$v_{CN} = K_1 e^{(-\alpha+j\beta)t} + K_2 e^{(-\alpha-j\beta)t}$$

$$v_{CN} = Ae^{-\alpha t} \cos(\beta t + \phi)$$

Exponentially Damped Sinusoid

Circuit is Underdamped



## SERIES RLC CIRCUITS

$$v_{CN} = e^{-\alpha t} (B_1 \cos(\beta t) + B_2 \sin(\beta t))$$

$$v_{CN} = e^{-\alpha t} \left[ \sqrt{B_1^2 + B_2^2} \cos\left(\beta t + \tan^{-1} \frac{B_2}{B_1}\right) \right]$$

Where:

$$A = \sqrt{B_1^2 + B_2^2}; \quad \phi = \tan^{-1} \frac{B_2}{B_1}$$

$$v_{CN} = Ae^{-\alpha t} \cos(\beta t + \phi) \text{ Exponentially Damped Sinusoid}$$



## SERIES RLC CIRCUITS

$$\alpha = \frac{R}{2L}; \quad \omega_0^2 = \frac{1}{LC} = \frac{1}{10^{-4}(10^{-6})} = 1 \times 10^{10}$$

R	$\alpha$	$\alpha^2$
30 Ω	$1.5 \times 10^5$	$2.25 \times 10^{10}$
20 Ω	$1.0 \times 10^5$	$1.0 \times 10^{10}$
10 Ω	$.5 \times 10^5$	$.25 \times 10^{10}$



## SERIES RLC CIRCUITS

$$v_{CN} = K_1 e^{(-\alpha+j\beta)t} + K_2 e^{(-\alpha-j\beta)t} \quad (@ t=0: v_{CN} = K_1 + K_2 \text{ must be Real})$$

$$v_{CN} = e^{-\alpha t} \left[ \boxed{B_1} \left[ \frac{e^{j\beta t} + e^{-j\beta t}}{2} \right] + \boxed{B_2} \left[ \frac{e^{j\beta t} + e^{-j\beta t}}{2j} \right] \right]$$

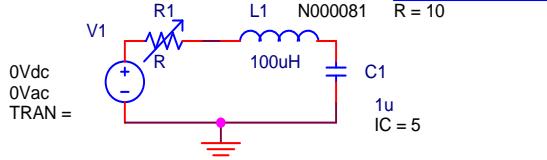
$$v_{CN} = e^{-\alpha t} \left( B_1 \left[ \frac{e^{j\beta t} + e^{-j\beta t}}{2} \right] + B_2 \left[ \frac{e^{j\beta t} + e^{-j\beta t}}{2j} \right] \right)$$

$$v_{CN} = e^{-\alpha t} (B_1 \cos(\beta t) + B_2 \sin(\beta t))$$



## SERIES RLC CIRCUITS

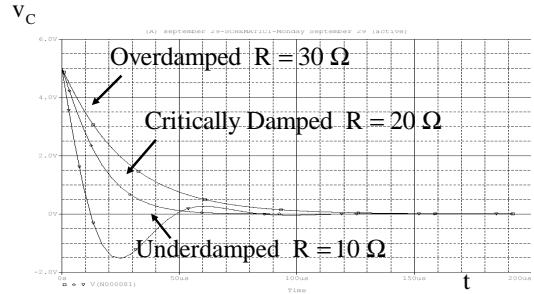
PARAMETERS:



Zero Input Response

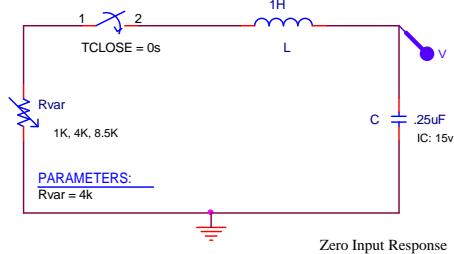


## SERIES RLC CIRCUITS

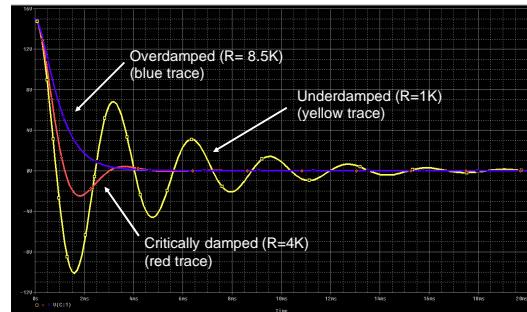


## SERIES RLC CIRCUITS

Example:



## SERIES RLC CIRCUITS



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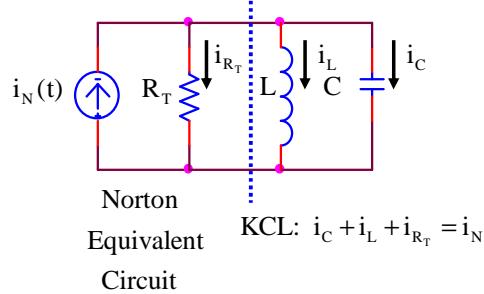
Lecture 10.3



## LECTURE 10.3 AGENDA

- Parallel RLC circuits
- 2<sup>nd</sup> order step response

### PARALLEL RLC CIRCUITS

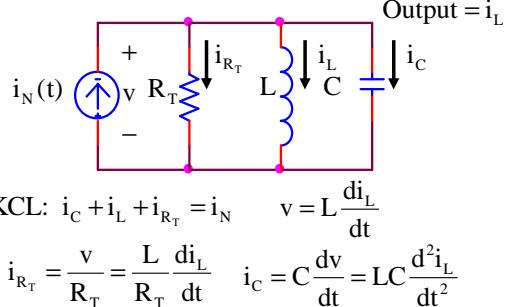


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### PARALLEL RLC CIRCUITS

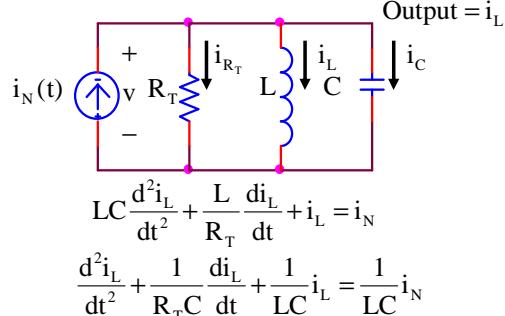


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### PARALLEL RLC CIRCUITS

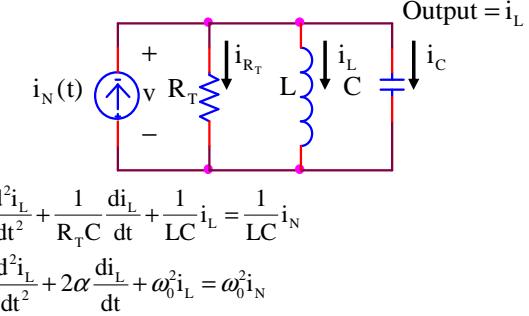


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### PARALLEL RLC CIRCUITS

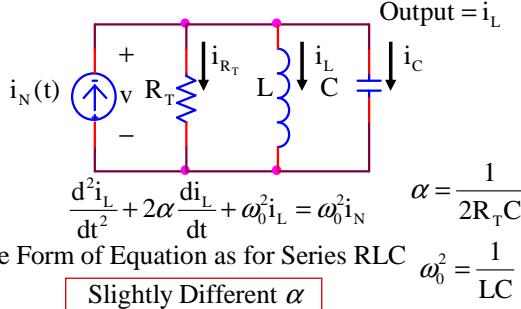


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## PARALLEL RLC CIRCUITS



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## PARALLEL RLC CIRCUITS

Parallel RLC Circuits

LHS of Differential Equation is Same for Any Output

Natural Response for Any Output

$$\frac{d^2y_N}{dt^2} + 2\alpha \frac{dy_N}{dt} + \omega_0^2 y_N = 0$$

Same as for Series RLC Circuits



## PARALLEL RLC CIRCUITS

Natural Response

$$\frac{d^2y_N}{dt^2} + 2\alpha \frac{dy_N}{dt} + \omega_0^2 y_N = 0$$

Characteristic Equation

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Same Roots as for Series RLC

Overdamping, Critical Damping, Underdamping

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## 2<sup>ND</sup> ORDER STEP RESPONSE

Any 2<sup>nd</sup> Order Circuit

Input = Au(t) = Step Function

Equation Relating Output to Input

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = Au(t)$$

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## 2<sup>ND</sup> ORDER STEP RESPONSE

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = Au(t)$$

$$\frac{d^2y}{dt^2} + \frac{a_1}{a_2} \frac{dy}{dt} + \frac{a_0}{a_2} y = \frac{A}{a_2} u(t)$$

Can Always Define  $\alpha = \frac{a_1}{2a_2}$ ;  $\omega_0^2 = \frac{a_0}{a_2}$

$$\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y = \frac{A}{a_2} u(t)$$

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## 2<sup>ND</sup> ORDER STEP RESPONSE

$$\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y = \frac{A}{a_2} u(t)$$

Natural Response

Characteristic Equation

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Roots will always be at  $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

Overdamping, Critical Damping, Underdamping

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## 2<sup>ND</sup> ORDER CIRCUITS

The Differential Equation for ANY Output for ANY 2<sup>nd</sup> Order Linear Circuit Can be Written as:

$$\frac{d^2y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_0^2 y = f[x(t)]$$

Characteristic Equation will Always be:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Natural Response Will Always Be:

Overdamped, Critically Damped, OR Underdamped

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## 2<sup>ND</sup> ORDER STEP RESPONSE

$$\text{We Defined } 2\alpha = \frac{a_1}{a_2}; \quad \omega_0^2 = \frac{a_0}{a_2}$$

$$[\alpha] = (\text{seconds})^{-1}; \quad [\omega_0] = (\text{seconds})^{-1}$$

$\omega_0$  = Undamped Natural Frequency

$$\text{Could Also Define } 2\zeta\omega_0 = \frac{a_1}{a_2}; \quad \omega_0^2 = \frac{a_0}{a_2}$$

$\zeta$  = Damping Ratio

$\zeta$  has no units

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## 2<sup>ND</sup> ORDER STEP RESPONSE

Roots of Characteristic Equation

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1, s_2 = \omega_0(-\zeta \pm \sqrt{(\zeta^2 - 1)})$$

$\zeta > 1 \Rightarrow$  Overdamped

$\zeta = 1 \Rightarrow$  Critically Damped

$\zeta < 1 \Rightarrow$  Underdamped

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## 2<sup>ND</sup> ORDER STEP RESPONSE

$$y(t) = y_N + y_F$$

$y_N \Rightarrow$  Overdamped,  
Critically Damped,  
or Underdamped

For Step Input =  $x(t) = Au(t)$

$$y_F = \frac{A}{a_0} = \text{Constant}$$

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## 2<sup>ND</sup> ORDER STEP RESPONSE

Step Response Provides Best Way to "See"  $y_N$

Input = Step

Output =  $y_N + y_F = y_N + \text{Constant}$

Time Behavior of Output is Determined by  $y_N$

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