ELECTRIC CIRCUITS
ECSE-2010

Lecture 13.1

LECTURE 13.1 AGENDA

- Partial fraction expansion (m<n)
  - 3 types of poles
    - Simple Real poles
    - Real Equal poles
    - Complex conjugate poles

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PARTIAL FRACTION EXPANSION

- Method for finding *f(t) from F(s) without taking the inverse laplace transform* (without integration)
- Concept: Expand F(s) into a sum of simple terms whose inverse laplace transforms we know…
  - Can then use tables
  - Use linearity property

PARTIAL FRACTION EXPANSION

Expand $F(s) = F_1(s) + F_2(s) + F_3(s) + ...$

From Linearity Property:

$$=> f(t) = \mathcal{L}^{-1}\{F(s)\} = f_1(t) + f_2(t) + f_3(t) + ...$$

$$\left.\begin{aligned} \text{Where } \mathcal{L}^{-1}\{F_1(s)\} &= f_1(t) \\ \mathcal{L}^{-1}\{F_2(s)\} &= f_2(t) \end{aligned}\right\} \text{Find } f_1(t),\ f_2(t),\ \text{etc. From Tables}$$

etc.

PARTIAL FRACTION EXPANSION

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}$$

$$F(s) = K\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

Factors of $N(s) = z_i$ => Zeros of F(s)

Factors of $D(s) = p_j$ => Poles of F(s)

PARTIAL FRACTION EXPANSION

There are only 3 Types of Poles:

Simple, Real Poles: $(s-4)$, $=> p_1 = 4$

Real, Equal Poles: $(s+3)^2$, $=> p_1 = p_2 = -3$

Complex Conjugate Poles: $(s^2 + 8s + 25)$

$$=> p_1, p_2 = -4 \pm j3$$

PARTIAL FRACTION EXPANSION

- There is a different way of doing Partial Fraction Expansion for Each Type of Pole
- Let's First Look at Simple Real Poles
  Then Complex Conjugate Poles
  Finally, Real, Equal Poles (Multiple Poles)
- Will First Look at Circuits where m < n:

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PARTIAL FRACTION EXPANSION

For m < n:

- Simple Real Poles

$$\text{Example: } F(s) = \frac{N(s)}{D(s)} = \frac{2s}{(s+3)(s+4)}$$

Simple Real Poles at $p_1 = -3,\ p_2 = -4$

To Find $\mathcal{L}^{-1}\{F(s)\}$ for Simple, Real Poles:

$$\text{Expand: } F(s) = \frac{A_1}{s+3} + \frac{A_2}{s+4}$$

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PARTIAL FRACTION EXPANSION

For m < n:

- Simple Real Poles $\quad F(s) = \dfrac{2s}{(s+3)(s+4)}$

$$\text{Expand: } F(s) = \frac{A_1}{s+3} + \frac{A_2}{s+4}$$

Know that $\mathcal{L}^{-1}[\dfrac{1}{s+\alpha}] = e^{-\alpha t}$

$$=> \ f(t) = A_1 e^{-3t} + A_2 e^{-4t}; \quad t \geq 0$$

Need Only to Find A_1 and A_2

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PARTIAL FRACTION EXPANSION

For m < n:

- Simple Real Poles

In General:

$$A_n = (s - p_n)F(s)\Big|_{s=p_n}$$

"Cover-Up Rule"

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PARTIAL FRACTION EXPANSION

For m < n:

- Simple Real Poles $\quad F(s) = \dfrac{2s}{(s+3)(s+4)}$

$$\text{Expand: } F(s) = \frac{A_1}{s+3} + \frac{A_2}{s+4}$$

To Find Coefficients, ...; Use "Cover-Up Rule":

$$A_1 = [(s - p_1)F(s)]\Big|_{s=p_1} \qquad p_1 = -3$$

$$A_2 = [(s - p_2)F(s)]\Big|_{s=p_2} \qquad p_2 = -4$$

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PARTIAL FRACTION EXPANSION

For m < n:

- Simple Real Poles $\quad F(s) = \dfrac{2s}{(s+3)(s+4)}$

$$\text{Expand: } F(s) = \frac{A_1}{s+3} + \frac{A_2}{s+4}$$

$$A_1 = [(s - p_1)F(s)]\Big|_{s=p_1}$$

$$=> A_1 = [(s+3)\frac{2s}{(s+3)(s+4)}]\Big|_{s=-3}$$

$$=> A_1 = [\frac{2s}{(s+4)}]\Big|_{s=-3} = \frac{2(-3)}{-3+4} = -6$$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Simple Real Poles $\quad F(s) = \dfrac{2s}{(s+3)(s+4)}$

Expand: $F(s) = \dfrac{A_1}{s+3} + \dfrac{A_2}{s+4}$

$$A_2 = [(s-p_2)F(s)]\Big|_{s=p_2}$$

$$\Rightarrow A_2 = [(s+4)\frac{2s}{(s+3)(s+4)}]\Big|_{s=-4}$$

$$\Rightarrow A_2 = [\frac{2s}{(s+3)}]\Big|_{s=-4} = \frac{2(-4)}{-4+3} = +8$$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Simple Real Poles $\quad F(s) = \dfrac{2s}{(s+3)(s+4)}$

Expand: $F(s) = \dfrac{A_1}{s+3} + \dfrac{A_2}{s+4}$

$$\Rightarrow A_1 = -6, \quad A_2 = +8$$

f(t) for Simple Poles $= A_1 e^{p_1 t} + A_2 e^{p_2 t}; \quad t \geq 0$

$$\Rightarrow f(t) = -6e^{-3t} + 8e^{-4t}; \quad t \geq 0$$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Simple Real Poles

In General:

Expand: $F(s) = \dfrac{A_1}{s-p_1} + \dfrac{A_2}{s-p_2} + \dfrac{A_3}{s-p_3} + \ .....$

$A_n = [(s-p_n)F(s)]\Big|_{s=p_n}$; $\quad$ Cover-Up Rule

$$\Rightarrow f(t) =)A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_3 e^{p_3 t} + \ .....) \quad t \geq 0$$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Complex Conjugate Poles

Example: $F(s) = \dfrac{2s}{s^2+8s+25} = \dfrac{2s}{s^2+2\alpha s+\omega_0^2}$

$$\alpha = 4, \ \omega_0^2 = 25, \ \beta = \sqrt{25-16} = 3;$$

Complex Conjugate Poles at $p_1, p_2 = -4 \pm j3$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Complex Conjugate Poles $\quad F(s) = \dfrac{2s}{s^2+8s+25}$

Complex Conjugate Poles at $p_1, p_2 = -4 \pm j3$

To find $\mathcal{L}^{-1}\{F(s)\}$ for Complex Conjugate Poles:

$$\text{Expand } F(s) = \frac{A}{s-(-4+j3)} + \frac{A^*}{s-(-4-j3)} = \frac{A}{s+4-j3} + \frac{A^*}{s+4+j3}$$

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PARTIAL FRACTION EXPANSION

$$\text{Expand } F(s) = \frac{A}{s+4-j3} + \frac{A^*}{s+4+j3}$$

$A =$ Complex Coefficient $= A_r + jA_i \ = |A| \underline{/\phi}$

$A^* =$ Complex Conjugate of $A = A_r - jA_i = |A| \underline{/-\phi}$

Let's Look at a Picture

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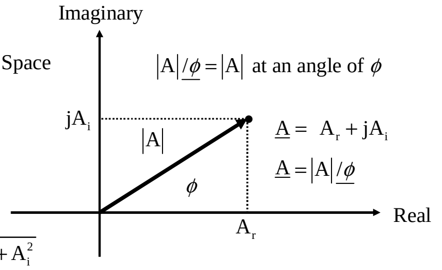
PARTIAL FRACTION EXPANSION

Complex Space

$|A|\underline{/\phi} = |A|$ at an angle of ϕ



$\underline{A} = A_r + jA_i$

$\underline{A} = |A|\underline{/\phi}$

$|A| = \sqrt{A_r^2 + A_i^2}$

$\phi = \tan^{-1}\dfrac{A_i}{A_r}$

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PARTIAL FRACTION EXPANSION

$$\text{Expand } F(s) = \frac{A}{s+4-j3} + \frac{A^*}{s+4+j3}$$

$A = \text{Complex Coefficient} = A_r + jA_i = |A|\underline{/\phi}$

$A^* = \text{Complex Conjugate of } A = A_r - jA_i = |A|\underline{/-\phi}$

$=> \; f(t) \;=\; Ae^{(-4+j3)t} + A^* e^{(-4-j3)t} \quad t \ge 0$

$=> \; f(t) \;=\; 2|A|e^{-4t}\cos(3t+\phi) \quad t \ge 0$

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PARTIAL FRACTION EXPANSION

$$\text{Expand } F(s) = \frac{A}{s+4-j3} + \frac{A^*}{s+4+j3}$$

To Find Coefficients, Use "Cover-Up Rule":

$$A = [(s-p_1)F(s)]\big|_{s=p_1}$$

$$A^* = [(s-p_2)F(s)]\big|_{s=p_2}$$

Good News: Need to Find Only 1

Bad News: Must use Complex Algebra

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PARTIAL FRACTION EXPANSION

$$F(s) = \frac{2s}{s^2+8s+25} = \frac{A}{(s+4-j3)} + \frac{A^*}{(s+4+j3)}$$

$$A = [(s+4-j3)\frac{2s}{(s+4-j3)(s+4+j3)}]\Big|_{s=-4+j3}$$

$$A = \frac{2(-4+j3)}{(-4+j3+4+j3)} = \frac{-8+j6}{j6}\left(\frac{j}{j}\right) = \frac{-j8-6}{-6}$$

$$\boxed{A = \frac{6+j8}{6} \qquad A^* = \frac{6-j8}{6}}$$

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PARTIAL FRACTION EXPANSION

$$A = \frac{6+j8}{6} = A_r + jA_i$$

$$\boxed{|A| = \sqrt{\frac{36+64}{36}} = \frac{10}{6} \qquad \phi = \tan^{-1}\frac{8}{6} = 51.3^0}$$

$=> \; f(t) \;=\; 2|A|e^{-\alpha t}\cos(\beta t+\phi)$

$$\boxed{\alpha = 4; \quad \beta = 3}$$

$=> \; f(t) = \dfrac{10}{3}e^{-4t}\cos(3t+51.3^0) \quad t \ge 0$

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PARTIAL FRACTION EXPANSION

In General:

$$\text{Expand } F(s) = \frac{A_1}{s-p_1} + \;....+ \frac{A}{s+\alpha-j\beta} + \frac{A^*}{s+\alpha+j\beta}$$

Find A_1 and $A = |A|\underline{/\phi}$ from Cover-Up Rule

$=> \; f(t) = A_1 e^{p_1 t} + \;....+ 2|A|e^{-\alpha t}\cos(\beta t+\phi) \quad t \ge 0$

Simple Poles   Complex Poles

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles: $p_1,\ p_2 = -\alpha$

Example: $F(s) = \dfrac{2s}{(s+3)^2}$

Real, Equal Poles at $p_1, p_2 = -3$

To Find $\mathcal{L}^{-1}\{F(s)\}$ for Real, Equal Poles:

Expand: $F(s) = \dfrac{A_1}{s+3} + \dfrac{A_2}{(s+3)^2}$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles: $F(s) = \dfrac{2s}{(s+3)^2}$

Expand: $F(s) = \dfrac{A_1}{s+3} + \dfrac{A_2}{(s+3)^2}$

$\Rightarrow\ f(t) = A_1 e^{-3t} + A_2 t e^{-3t} \quad t \geq 0$

Need Only to Find A_1 and A_2

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles: $F(s) = \dfrac{2s}{(s+3)^2}$

To Find A_2, Use Cover-Up Rule:

$\Rightarrow \quad A_2 = [(s+3)^2 F(s)]\big|_{s=-3}$

$A_2 = 2s\big|_{s=-3} = -6$

Cannot Use Cover-Up Rule for A_1

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles: $F(s) = \dfrac{2s}{(s+3)^2}$

Expand: $F(s) = \dfrac{A_1}{s+3} + \dfrac{A_2}{(s+3)^2}$

$F(0) = \dfrac{2(0)}{(0+3)^2} = 0 = \dfrac{A_1}{0+3} + \dfrac{-6}{(0+3)^2} \quad \Rightarrow A_1 = 2$

$f(t) = 2e^{-3t} - 6te^{-3t} \quad t \geq 0$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles – Double Pole:

Expand $F(s) = \dfrac{A_1}{s-p_1} + \ ..+[\dfrac{A_{n1}}{s-p_n} + \dfrac{A_{n2}}{(s-p_n)^2}]$

$A_{n2} = \left[(s-p_n)^2 F(s)\right]\big|_{s=p_n}$; Cover-Up Rule

Usually Find A_{n1} from evaluating $F(0)$ or $F(1)$

$\Rightarrow f(t) = (\underbrace{A_1 e^{p_1 t} + \ ....}_{\text{Simple Poles}} + \underbrace{A_{n1} e^{p_n t} + A_{n2} t e^{p_n t}}_{\text{Repeated Poles}}) \quad t \geq 0$

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PARTIAL FRACTION EXPANSION

For $m < n$:

- Real, Equal Poles – Double Pole:

Can Also Use Differentiation:

$A_{n2} = \left[(s-p_n)^2 F(s)\right]\big|_{s=p_n}$; Cover-Up Rule

$A_{n1} = \dfrac{d}{ds}\left[(s-p_n)^2 F(s)\right]\Big|_{s=p_n}$; Differentiation

$\Rightarrow f(t) = (\underbrace{A_1 e^{p_1 t} + \ ....}_{\text{Simple Poles}} + \underbrace{A_{n1} e^{p_n t} + A_{n2} t e^{p_n t}}_{\text{Repeated Poles}}) \quad t \geq 0$

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PARTIAL FRACTION EXPANSION

- Real, Equal Poles – Triple Pole:

Expand $F(s) = \frac{A_1}{s-p_1} + \ ..+[\frac{A_{n1}}{s-p_n} + \frac{A_{n2}}{(s-p_n)^2} + \frac{A_{n3}}{(s-p_n)^3}]$

$A_{n3} = \left[(s-p_n)^3 F(s)\right]\Big|_{s=p_n}$; Cover-Up Rule

Usually Find A_{n2} from F(0) or F(1)

Usually Find A_{n1} from $\lim_{s\to\infty} sF(s)$

=> $\boxed{f(t) = (A_1 e^{p_1 t} + \ ....+ A_{n1}e^{p_n t} + A_{n2}te^{p_n t} + \frac{1}{2!}\ A_{n3}t^2 e^{p_n t})}$ $t \geq 0$

PARTIAL FRACTION EXPANSION

- Real, Equal Poles – Triple Pole: Using Differentiation

$A_{n3} = \left[(s-p_n)^3 F(s)\right]\Big|_{s=p_n}$; Cover-Up Rule

$A_{n2} = \frac{d}{ds}[(s-p_n)^3 F(s)]\Big|_{s=p_n}$; Differentiation

$A_{n1} = \frac{1}{2!}\frac{d^2}{ds^2}[(s-p_n)^3 F(s)]\Big|_{s=p_n}$; Differentiation

=> $f(t) = (A_1 e^{p_1 t} + \ ....+ A_{n1}e^{p_n t} + A_{n2}te^{p_n t} + \frac{1}{2!}\ A_{n3}t^2 e^{p_n t})$ $t \geq 0$

PARTIAL FRACTION EXPANSION

- What happens when m = n?

$$F(s) = K\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \qquad m = n$$

- Use Long Division

$$F(s) = K + \text{Remainder}$$

Remainder will have m < n

PARTIAL FRACTION EXPANSION

$$f(t) = \mathcal{L}^{-1}\{K\} + \mathcal{L}^{-1}\{\text{Remainder}\}$$

Use Partial Fraction Expansion to Find $\mathcal{L}^{-1}\{\text{Remainder}\}$

$$\mathcal{L}^{-1}\{K\} = K\delta(t)$$

$$f(t) = K\delta(t) + \mathcal{L}^{-1}\{\text{Remainder}\}$$

Most Circuits will have m < n

CIRCUITS WITH LAPLACE

- Will Do in 2 Steps:
- Method 1
  - First Find Differential Equation
  - Transform to an Algebraic Equation
  - Take Inverse Laplace to Find y(t)
- Method 2
  - Define s-domain Circuits
  - No More Differential Equations!

CIRCUITS WITH LAPLACE

- Same Result as Solving Differential Equation:
  - Not Clear that this is Easier for 1st Order Circuits with Switched DC Inputs
  - Still have to find Differential Equation
- Advantage?:
  - Can now Solve Circuits of ANY Order
  - Can now Solve Circuits with ANY Input
  - IF we can find the Differential Equation

CIRCUITS WITH LAPLACE

- Let's Practice with Activity 20-2:
- This is a 2^{nd} Order Circuit:
  - Need 2^{nd} Order Differential Equation
  - Ignore Problem Statement about "s-domain diagram" and "initial condition sources"
  - Ignore "Series-Parallel Impedance Reduction"

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