ELECTRIC CIRCUITS
ECSE-2010

Lecture 14.1

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LECTURE 14.1 AGENDA

- Example Problems
- Initial and final values
- Circuit elements in the S-domain

CIRCUITS WITH LAPLACE

- Will Do in 2 Steps:
- Method 1
  - First find differential equation
  - Transform to an algebraic equation
  - Take inverse laplace to find y(t)
- Method 2
  - Define s-domain circuits
  - No more differential equations!

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CIRCUITS WITH LAPLACE

- Same result as solving differential equation:
  - Not clear that this is easier for 1st order circuits with switched DC inputs
  - Still have to find differential equation
- Advantage?:
  - Can now solve circuits of ANY order
  - Can now solve circuits with ANY input
  - IF we can find the differential equation

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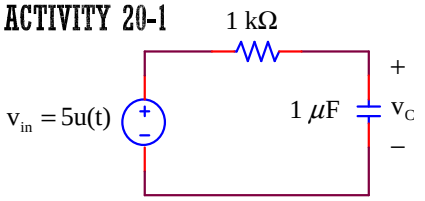
ACTIVITY 20-1



First Order Circuit

Switched DC Input

$v_{C0} = 0$ V $\quad v_{CSS} = 5$ V

$v_C(t) = v_{CSS} + (v_{C0} - v_{CSS})e^{-(t-t_0)/\tau}$

$\tau = RC = 1$ msec $\quad v_C(t) = 5(1 - e^{-t/\tau}) \quad t \geq 0$

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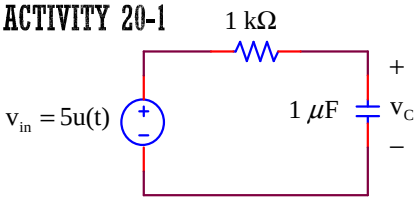
ACTIVITY 20-1



Find Differential Equation

Relating $v_C(t)$ to v_{in}

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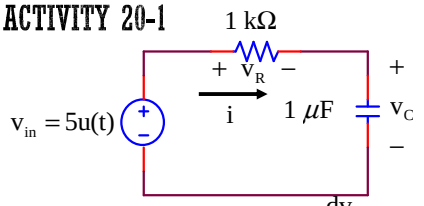

$$v_C + v_R = v_{in} \qquad v_R = iR \qquad i = C\frac{dv_C}{dt}$$

$$v_C + RC\frac{dv_C}{dt} = v_{in} \qquad \frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{v_{in}}{RC} \qquad v_C(0^+) = 0$$

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$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{v_{in}}{RC} \qquad v_{in} = 5u(t)$$

$$L\left\{\frac{dv_C}{dt} + \frac{v_C}{RC}\right\} = L\left\{\frac{5u(t)}{RC}\right\}$$

$$\left(sV_C(s) - v_C(0^-)\right) + \frac{V_C(s)}{RC} = \frac{5}{RC}\frac{1}{s}$$

$$\left(sV_C(s) - v_{C0}\right) + \frac{V_C(s)}{RC} = \frac{5}{sRC}$$

Algebraic Equation

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$$\left(sV_C(s) - v_{C0}\right) + \frac{V_C(s)}{RC} = \frac{5}{sRC}$$

$$V_C(s)\left(s + \frac{1}{RC}\right) = \frac{5}{sRC} + v_{C0}$$

$$V_C(s) = \frac{5/RC}{s(s+1/RC)} + \frac{v_{C0}}{(s+1/RC)}$$

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$$V_C(s) = \frac{5/RC}{s(s+1/RC)} + \frac{v_{C0}}{(s+1/RC)}$$

$$\frac{5/RC}{s(s+1/RC)} = \frac{A_1}{s} + \frac{A_2}{(s+1/RC)}$$

$$A_1 = \left.\frac{5/RC}{(s+1/RC)}\right|_{s=0} = 5$$

$$A_2 = \left.\frac{5/RC}{s}\right|_{s=-\frac{1}{RC}} = -5$$

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$$V_C(s) = \frac{5}{s} + \frac{-5}{s+1/RC} + \frac{v_{C0}}{(s+1/RC)}$$

$$v_C(t) = 5 - 5e^{-t/RC} + v_{C0}e^{-t/RC}$$

$$v_C(t) = 5 + (v_{C0} - 5)e^{-t/RC}$$

$$v_{C0} = 0; \quad RC = \tau$$

$$v_C(t) = 5(1 - e^{-t/\tau}) \text{ volts} \qquad t \geq 0$$

Same as Before

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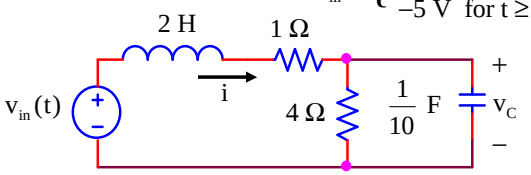
$$v_{in} = \begin{cases} +5\text{ V} & \text{for } t < 0 \\ -5\text{ V} & \text{for } t \geq 0 \end{cases}$$



Find I(s), $V_C(s)$

You do have to find $v_C(t)$

(e.g. if you like, use Maple or other methods)

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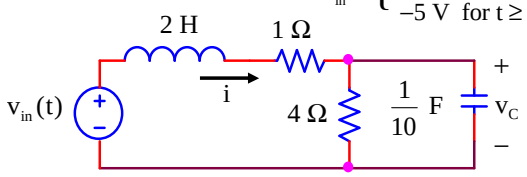
ACTIVITY 20-2

$$v_{in} = \begin{cases} +5\text{ V} & \text{for } t < 0 \\ -5\text{ V} & \text{for } t \geq 0 \end{cases}$$



First: Find $i_L(0^-)$, $v_C(0^-)$ Initial Conditions

Second: Find Differential Equation for v_C

Third: Solve for $V_C(s)$, $I(s)$

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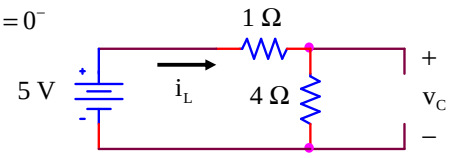
ACTIVITY 20-2

$t = 0^-$



$i_L(0^-) = 1\text{ A}$

$v_C(0^-) = 4\text{ V}$

$$i_C(0^-) = 0 = \frac{1}{10}\frac{dv_C}{dt}(0^-)$$

$$\Rightarrow \frac{dv_C}{dt}(0^-) = 0$$

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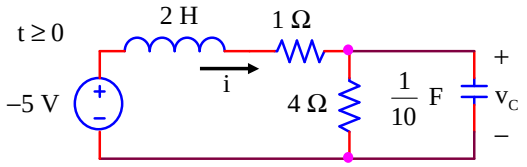
$t \geq 0$



KVL: $v_C + 1(i) + 2\frac{di}{dt} = -5$ KCL: $i = \frac{v_C}{4} + \frac{1}{10}\frac{dv_C}{dt}$

$$v_C + \left(\frac{v_C}{4} + \frac{1}{10}\frac{dv_C}{dt}\right) + 2\left(\frac{1}{4}\frac{dv_C}{dt} + \frac{1}{10}\frac{d^2v_C}{dt^2}\right) = -5$$

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$$v_C + \left(\frac{v_C}{4} + \frac{1}{10}\frac{dv_C}{dt}\right) + 2\left(\frac{1}{4}\frac{dv_C}{dt} + \frac{1}{10}\frac{d^2v_C}{dt^2}\right) = -5$$

$$\frac{1}{5}\frac{d^2v_C}{dt^2} + \frac{6}{10}\frac{dv_C}{dt} + \frac{5}{4}v_C = -5$$

$$\frac{d^2v_C}{dt^2} + 3\frac{dv_C}{dt} + \frac{25}{4}v_C = -25$$

$$4\frac{d^2v_C}{dt^2} + 12\frac{dv_C}{dt} + 25v_C = -100$$

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$$4\frac{d^2v_C}{dt^2} + 12\frac{dv_C}{dt} + 25v_C = -100$$

$$L\left(\frac{d^2v_C}{dt^2}\right) = s^2V_C(s) - \underbrace{sv_C(0^-)}_{4s} - \underbrace{\frac{dv_C}{dt}(0^-)}_{0}$$

$$4\left(s^2V_C(s) - 4s\right) + 12\left(sV_C(s) - 4\right) + 25V_C(s) = -\frac{100}{s}$$

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$$4\left(s^2V_C(s) - 4s\right) + 12\left(sV_C(s) - 4\right) + 25V_C(s) = -\frac{100}{s}$$

$$V_C(s)\left(4s^2 + 12s + 25\right) = -\frac{100}{s} + 16s + 48$$

$$V_C(s)\left(4s^2 + 12s + 25\right) = \frac{-100 + 16s^2 + 48s}{s}$$

$$\boxed{V_C(s) = \frac{16s^2 + 48s - 100}{s(4s^2 + 12s + 25)}}$$

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ACTIVITY 20-2

$$V_C(s) = \frac{16s^2 + 48s - 100}{s(4s^2 + 12s + 25)}$$

$$\text{KCL: } i = \frac{v_C}{4} + \frac{1}{10}\frac{dv_C}{dt}$$

$$L(i) = L\left(\frac{v_C}{4} + \frac{1}{10}\frac{dv_C}{dt}\right)$$

$$I(s) = \frac{V_C(s)}{4} + \frac{1}{10}\left(sV_C(s) - 4\right)$$

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ACTIVITY 20-2

$$I(s) = \frac{V_C(s)}{4} + \frac{1}{10}\left(sV_C(s) - 4\right)$$

Lots of Algebra Later...

$$I(s) = \frac{4s^2 - 8s - 25}{4s^3 + 12s^2 + 25s}$$

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INITIAL AND FINAL VALUES

- Once we have found F(s), the initial and final values of f(t) can often be found <u>without</u> finding the inverse laplace transform:
  - Initial Value Theorem
  - Final Value Theorem

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INITIAL VALUE THEOREM

$$\boxed{f(t = 0^+) = f_0 = \lim_{s\to\infty} sF(s)}$$

=> Can Find Initial Value Directly from F(s)

F(s) must be a Proper Rational Fraction

$(m < n)$

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INITIAL VALUE THEOREM

Example:

$$F(s) = \frac{2s}{(s^2 + 7s + 12)}$$

$$\Rightarrow f(0^+) = f_0 = \lim_{s\to\infty} sF(s)$$

$$\Rightarrow f(0^+) = \lim_{s\to\infty} \frac{2s^2}{(s^2 + 7s + 12)} = 2$$

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FINAL VALUE THEOREM

$$\boxed{f(t \to \infty) = f_{ss} = \lim_{s\to 0} sF(s)}$$

=> Can Find Final Value Directly from Y(s)
Limit Must Exist

=> Cannot Use When:

Multiple Poles at $s = 0$

Poles on Imaginary Axis

Poles in Right Half of s-plane

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FINAL VALUE THEOREM

Example:

$$F(s) = \frac{2s}{(s^2 + 7s + 12)}$$

$\Rightarrow f(t \to \infty) = f_{ss}$

$$\Rightarrow f(t \to \infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{2s^2}{(s^2 + 7s + 12)} = 0$$

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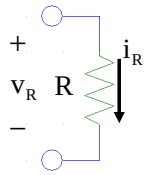
CIRCUIT ELEMENTS IN S-DOMAIN



Time Domain

- Time Domain:

  $v_R(t) = i_R(t)R$; Algebraic Equation
- s-Domain:

  $L\{v_R(t)\} = L\{i_R(t)R\}$

  $V_R(s) = I_R(s)\,R$ => Still Algebraic

  => Resistor still looks like a Resistor

  => Resistor has an "Impedance" of R Ω in s-Domain

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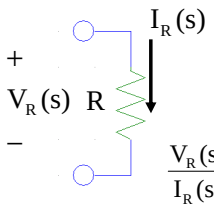
CIRCUIT ELEMENTS IN S-DOMAIN



s-Domain

- Impedance:

  $V_R(s) = I_R(s)\,R$

  $\frac{V_R(s)}{I_R(s)}$ has units of Ohms

  $\frac{V_R(s)}{I_R(s)} = Z_R(s) = \text{Impedance of Resistor}$

  $Z_R(s) = R\ \Omega$

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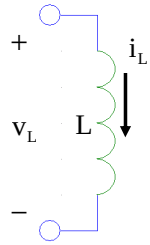
CIRCUIT ELEMENTS IN S-DOMAIN



Time Domain

- Time Domain:

  $v_L = L\frac{di_L}{dt}$; Differential Equation
- s-Domain:

  $L(v_L) = L(L\frac{di_L}{dt})$

  $\Rightarrow V_L(s) = L\{sI_L(s) - i_L(0^-)\}$

  => Algebraic Equation

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CIRCUIT ELEMENTS IN S-DOMAIN

$$V_L(s) = sL\,I_L(s) - Li_L(0^-)$$

In s-Domain:

Inductor looks like an "Impedance" of value sL Ω in Series with a Voltage Source of $-Li_L(0^-)$ Volts

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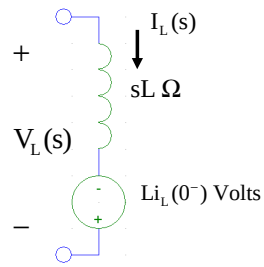
CIRCUIT ELEMENTS IN S-DOMAIN

Model for L in s-Domain



$$V_L(s) = sL\,I_L(s) - Li_L(0^-)$$

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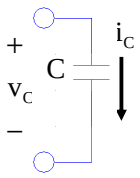
CIRCUIT ELEMENTS IN S-DOMAIN

Time Domain



- Time Domain:

  $i_C = C\dfrac{dv_C}{dt}$; Differential Equation

- s-Domain:

  $L(i_C) = L(C\dfrac{dv_C}{dt})$

  $\Rightarrow I_C(s) = C\,\{sV_C(s) - v_C(0^-)\}$

  $\Rightarrow V_C(s) = \dfrac{I_C(s)}{sC} + \dfrac{v_C(0^-)}{s}$

  $\Rightarrow$ Algebraic Equation

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CIRCUIT ELEMENTS IN S-DOMAIN

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

In s-Domain:

Capacitor looks like an "Impedance"

of value $\dfrac{1}{sC}\ \Omega$ in Series with

a Voltage Source of $\dfrac{v_C(0^-)}{s}$ Volts

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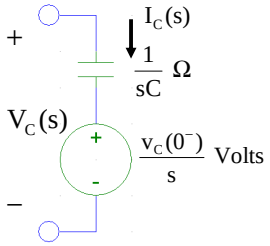
CIRCUIT ELEMENTS IN S-DOMAIN

Model for C in s-Domain



$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

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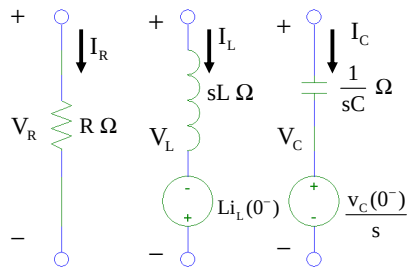
CIRCUIT ELEMENTS IN S-DOMAIN



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CIRCUIT ELEMENTS IN S-DOMAIN

For No Initial Stored Energy:

$v_C(0^-) = i_L(0^-) = 0$

Inductor has "Impedance" $Z_L = sL\ \Omega$

Capacitor has "Impedance" $Z_C = \dfrac{1}{sC}\ \Omega$

Resistor ALWAYS has "Impedance" $Z_R = R\ \Omega$

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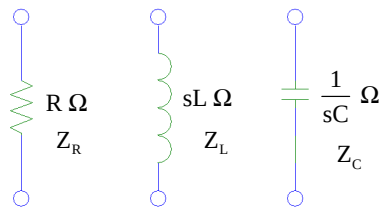
WITH NO INITIAL STORED ENERGY



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S-DOMAIN CIRCUIT ANALYSIS

- Input $= L\{x(t)\} = X(s)$
- Replace all i(t), v(t) $\rightarrow$ I(s), V(s)
- Use s-Domain Models for R, L, C
- Use all techniques developed for Resistive Circuits to find Output in s-Domain = Y(s)
- Take Inverse Laplace Transform of Y(s) to find Output y(t)
- We'll Practice with Activities 21-1 (today), & 22-1(Thursday)

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