1. (11 points) Let $x$ be a point chosen at random in the real interval $[0, 2]$. Now let

$A = \{ x \in [0,2] \mid x < 1 \}$, $B = \{ x \in [0,2] \mid x > 0.5 \}$, and $C = \{ x \in [0,2] \mid |x-1| < 0.25 \}$.

1.a. (3 points) Calculate $P[A], P[B], P[C]$.

Solution

Uniform on $[0,2]$ means that for $0 \leq a < b \leq 2$, we have

$$P[[a,b]] = \frac{b-a}{2-0} = \frac{b-a}{2}.$$ 

So we have $P[A] = P[[0,1]] = \frac{1-0}{2} = 0.5$, $P[B] = P[[0.5,2]] = \frac{2-0.5}{2} = 0.75$, and $P[C] = P[[0.75,1.25]] = \frac{1.25-0.75}{2} = 0.25$.

1.b. (5 points) Calculate $P[A\mid B], P[B\mid C], P[A\mid C], P[A\mid C^c]$, and $P[B\mid C^c]$. Clearly justify your answers.

Solution

Following from the definition of conditional probability, we have

$$P[A\mid B] = \frac{P[A \cap B]}{P[B]} = \frac{P[[0.5,1]]}{0.75} = \frac{0.25}{0.75} = \frac{1}{3}.$$ 

$$P[B\mid C] = \frac{P[B \cap C]}{P[C]} = \frac{P[[0.75,0.25]]}{0.25} = \frac{0.25}{0.25} = 1.$$ 

$$P[A\mid C] = \frac{P[A \cap C]}{P[C]} = \frac{P[[0.75,1]]}{0.25} = \frac{0.125}{0.25} = \frac{1}{2}.$$ 

$$P[A\mid C^c] = \frac{P[A \cap C^c]}{P[C^c]} = \frac{P[[0,0.75]]}{P[[0,0.75] \cup [1.25,2]]} = \frac{0.375}{0.375+0.375} = \frac{1}{2}.$$ 

$$P[B\mid C^c] = \frac{P[B \cap C^c]}{P[C^c]} = \frac{P[[0.5,0.75] \cup [1.25,2]]}{P[[0,0.75] \cup [1.25,2]]} = \frac{0.125+0.375}{0.75+0.375} = \frac{0.5}{3}.$$
1.c. (3 points) Using the results from parts 1.a and 1.b, show whether $A$ and $B$ are independent. Show whether $B$ and $C$ are independent. Show whether $A$ and $C$ are independent. Clearly justify your answer in each case.

Solution

$$P[A|B] = \frac{1}{3} \neq P[A] = 0.5, \text{ so } A \text{ and } B \text{ are not independent.}$$

$$P[B|C] = 1 \neq P[B] = 0.75, \text{ so } B \text{ and } C \text{ are not independent.}$$

$$P[A|C] = 0.5 = P[A], \text{ so } A \text{ and } C \text{ are independent.}$$

2. (6 points) ECSE buys half the computer chips it needs (for labs) from Company A, 40% from Company B, and 10% from Company C. After testing a large number of these chips, we find that 0.5% of the chips from Company A are defective, 1% of the chips from Company B are defective, and 0.1% from Company C are defective. Now we pick one chip at random from all the chips and find it defective. What is the probability that chip came from Company A? From Company B? From Company C?

Solution

From the Total Probability Theorem:


$$= 0.005(0.5) + 0.010(0.4) + 0.001(0.1) = 0.0066$$

From Bayes Theorem:

$$P[A|\text{def.}] = \frac{P[\text{def.}|A]P[A]}{P[\text{def.}]} = \frac{0.005(0.5)}{0.0066} = 0.3788$$

Similarly,

$$P[B|\text{def.}] = \frac{P[\text{def.}|B]P[B]}{P[\text{def.}]} = \frac{0.010(0.4)}{0.0066} = 0.6061$$

Similarly,

$$P[C|\text{def.}] = \frac{P[\text{def.}|C]P[C]}{P[\text{def.}]} = \frac{0.001(0.1)}{0.0066} = 0.015$$
3. (4 points) For any 2 events $A$ and $B$, show that if $A$ and $B$ are independent, then $A^C$ and $B$ are independent. You can assume that $P[B] \neq 0$.

Solution

If $A$ and $B$ are independent, then we know that $P[A|B] = P[A]$. We also know that for fixed $B$ with $P[B] \neq 0$, we showed in class that $P[\cdot|B]$ is a probability law and satisfies all the properties of a probability law. Thus, for any $A$ we have $P[A^c|B] = 1 - P[A|B]$. So if $A$ and $B$ are independent, then we have $P[A^c|B] = 1 - P[A|B] = 1 - P[A] = P[A^c]$. So $A$ and $B^c$ are independent.
4. (8 points) Following the Binary Channel Example in Slide set 4a, assume $\varepsilon = \varepsilon_0 = \varepsilon_1$. That is, the conditional probability of error is the same whether a 0 is sent or a 1 is sent. Our goal in this problem is to find the value of $\varepsilon$ that makes the output of the channel independent of the input.

4.a. (4 points) Using the definitions of $B_0 = \{(0,0),(0,1)\}$ and $B_1 = \{(1,0),(1,1)\}$ from slide 19, and letting $A_0 = \{(0,0),(1,0)\}$ and $A_1 = \{(0,1),(1,1)\}$, find the value of $\varepsilon$ for which $B_0$ and $A_i$ are independent. Justify your answer analytically. Note that $B_i$ represents the event of $i = 0,1$ being sent, while $A_i$ represents the event of $i = 0,1$ being received.

Solution

From slide 13, we have $\varepsilon = \varepsilon_0 = P\left[\{1\ \text{is rec'd}\}\mid \{0\ \text{was sent}\}\right] = P\left[A_i\mid B_0\right]$ and $\varepsilon = \varepsilon_1 = P\left[\{0\ \text{is rec'd}\}\mid \{1\ \text{was sent}\}\right] = P\left[A_i\mid B_1\right]$. Thus $P\left[A_i\mid B_i\right] = 1 - \varepsilon$.

Applying the Total Probability Theorem, we have

$$P[A_i] = P[A_i\mid B_0]P[B_0] + P[A_i\mid B_1]P[B_1]$$

$$= \varepsilon(1 - p) + (1 - \varepsilon)p.$$ 

We know that $B_0$ and $A_i$ are independent if and only if (iff) $P[A_i] = P[A_i\mid B_0]$. That is, iff $\varepsilon(1 - p) + (1 - \varepsilon)p = \varepsilon$ which happens iff $\varepsilon - p\varepsilon + p - \varepsilon p = \varepsilon$ iff $p = 2\varepsilon p$ iff $1 = 2\varepsilon$ iff $\varepsilon = 0.5$.

This makes intuitive sense since with a probability of error of one half, the output of the channel is equally likely to be a 1 or a 0 regardless of whether a 0 or 1 is transmitted.

4.b. (2 points) Now show that $B_0$ and $A_0$ are independent using your result from problem 3 and part a, and then do the same for $B_i$ and $A_0$ and for $B_i$ and $A_i$.

Solution

We showed in part a that $B_0$ and $A_i$ are independent for $\varepsilon = 0.5$. Noting that $A_i = A_i^c$, we use the result of Problem 3 to see that $B_0$ and $A_0$ are independent. Then noting that $B_i = B_i^c$, we see that $B_i$ and $A_0$ are independent. And finally, using $A_0 = A_i^c$, we see that $B_i$ and $A_i$ are independent.
4.c. (2 points) Finally, say whether the channel with this value of $\varepsilon$ can be used to transmit information. Justify your answer.

Solution

In this case, with $\varepsilon = 0.5$, the channel is useless because it is not possible to determine anything about the input from the output, so no information can be transmitted through this channel.