1. (12 points) Two distinct fair dice are tossed independently. We record the number on die 1 followed by the number on die 2 as an ordered pair, so that our sample space is

\[ S = \{ \zeta = (s_1, s_2) \mid s_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2 \}. \]

1.a. (3 points) Clearly define the probability law that corresponds to this problem given that the dice are fair and independent. Specifically, what is \( P[(i, j) \mid i = j] \)? Justify your answer.

Solution

Since the dice are independent, for any pair \( \zeta = (i, j) \in S \) with \( i, j \in \{1, 2, 3, 4, 5, 6\} \), we have

\[ P[(i, j)] = P\{i\}P\{j\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \]

Then because \( S \) is a discrete set, for any event \( A \subseteq S \), we have

\[ P[A] = \sum_{\zeta \in A} P\{\zeta\} \]

Finally,

\[ P[(i, j) \mid i = j] = P[(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)] \]

\[ = \sum_{i,j} P\{i,j\} = 6 \times \frac{1}{36} = \frac{1}{6}. \]

1.b. (3 points) Define the random variable \( X_1 \) on \( S \) by

\[ X_1((i, j)) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases} \]

Clearly identify range \( S_{X_1} = \{x_1, x_2, x_3, \ldots\} \subseteq \mathbb{R} \) for this random variable and the sets

\( A_0 = \{ \zeta \in S \mid X_1(\zeta) = 0 \} \) and \( A_i = \{ \zeta \in S \mid X_1(\zeta) = 1 \} \). Show that \( A_0 \) and \( A_i \) form a partition of \( S \).

Solution

Here \( S_{X_1} = \{0, 1\} \),
\[ A_0 = \{ \varepsilon \in S \mid X_1(\varepsilon) = 0 \} = \{(i, j) \in S \mid i < j \} \]
\[ = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6) \} \]

and \( A_1 = A_0^c \). We know that \( A_0 \) and \( A_1 \) form a partition of \( S \) since we know that any set and its complement form a partition of the space.

1.c. (3 points) For the random variable \( X_1 \) defined in (b), compute the probability mass function (PMF) \( p_{X_1}(x_k) \) for each \( x_k \in S_{X_1} \), directly from the \( A_k \) as shown in class and the book. Which of the well-known random variables covered in class and in the book is \( X_1 \)? Identify the value of any parameters associated with this well-known random variable. Justify your answer.

**Solution**

Using \( p_{X_1}(x_k) = P[X_1 = x_k] = P[A_k] \) and the fact that each element of \( A_k \) has probability \( \frac{1}{36} \), we have \( p_{X_1}(x_k) = P[A_k] = \frac{|A_k|}{36} \), where \( |A_k| \) is the number of elements in \( A_k \).

So, we have \( p_{X_1}(0) = P[A_0] = \frac{15}{36} \) and \( p_{X_1}(1) = P[A_1] = \frac{21}{36} \).

This is a Bernoulli random variable with \( p = \frac{21}{36} \).

1.d. (3 points) For the random variable \( X_1 \) defined in (b), compute \( E[X_1] \) using the definition given in Eq. (3.8) on page 104 in the book. Show your work.

**Solution**

\[
E[X_1] = \sum_{k=0}^{\infty} x_k p_{X_1}(x_k) = \sum_{k=0}^{1} k p_{X_1}(k) = 0 \left( \frac{15}{36} \right) + 1 \left( \frac{21}{36} \right) = \frac{21}{36}.
\]
2. (14 points) For the sample space defined in problem 1, define a new random variable \( X_2((i, j)) = i + j \).

2.a. (2 points) What is the range \( S_{X_2} = \{x_1, x_2, x_3, \ldots\} \subseteq \mathbb{R} \) for this random variable?

Solution

\[ S_{X_2} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \]

2.b. (4 points) For this random variable \( X_2 \), list the elements of the set \( A_k = \{\zeta \in S \mid X_2(\zeta) = x_k\} \) for each \( x_k \in S_{X_2} \), and show that the \( A_k \) together form a partition of \( S \).

Solution

Here, for \( k = 2, 3, \ldots, 12 \), \( A_k = \{(i, j) \in S \mid X_2((i, j)) = i + j = k\} \), so

\[
\begin{align*}
A_2 &= \{(1,1)\}, \\
A_3 &= \{(1,2), (2,1)\}, \\
A_4 &= \{(1,3), (3,1)(2,2)\}, \\
A_5 &= \{(1,4), (4,1), (2,3), (3,2)\}, \\
A_6 &= \{(1,5), (5,1), (2,4), (4,2), (3,3)\}, \\
A_7 &= \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}, \\
A_8 &= \{(2,6), (6,2), (3,5), (5,3), (4,4)\}, \\
A_9 &= \{(3,6), (6,3), (4,5), (5,4)\}, \\
A_{10} &= \{(4,6), (6,4), (5,5)\}, \\
A_{11} &= \{(5,6), (6,5)\}, \\
A_{12} &= \{(6,6)\}.
\end{align*}
\]

We can see that they are pairwise disjoint and their union is \( S \) by inspection.

2.c. (4 points) For the random variable \( X_2 \), compute the probability mass function (PMF) \( p_{X_2}(x_k) \) for each \( x_k \in S_{X_2} \), directly from the \( A_k \) as shown in class and the book. Justify your answer.
Solution

As in problem 1, using $p_{X}(x) = P[X = x] = P[A_k]$ and the fact that each element of $A_k$ has probability $\frac{1}{36}$, we have $p_{X}(x) = P[A_k] = \frac{|A_k|}{36}$, where $|A_k|$ is the number of elements in $A_k$.

So, we have

$$
\begin{align*}
p_{X_1}(2) &= P[A_2] = P[(1,1)] = \frac{1}{36}, \\
p_{X_1}(3) &= P[A_3] = P[(1,2),(2,1)] = \frac{2}{36}, \\
p_{X_1}(4) &= P[A_4] = P[(1,3),(3,1),(2,2)] = \frac{3}{36}, \\
p_{X_1}(5) &= P[A_5] = P[(1,4),(4,1),(2,3),(3,2)] = \frac{4}{36}, \\
p_{X_1}(6) &= P[A_6] = P[(1,5),(5,1),(2,4),(4,2),(3,3)] = \frac{5}{36}, \\
p_{X_1}(7) &= P[A_7] = P[(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)] = \frac{6}{36}, \\
p_{X_1}(8) &= P[A_8] = P[(2,6),(6,2),(3,5),(5,3),(4,4)] = \frac{7}{36}, \\
p_{X_1}(9) &= P[A_9] = P[(3,6),(6,3),(4,5),(5,4)] = \frac{8}{36}, \\
p_{X_1}(10) &= P[A_{10}] = P[(4,6),(6,4),(5,5)] = \frac{9}{36}, \\
p_{X_1}(11) &= P[A_{11}] = P[(5,6),(6,5)] = \frac{10}{36}, \\
p_{X_1}(12) &= P[A_{12}] = P[(6,6)] = \frac{11}{36}.
\end{align*}
$$

2.d. (4 points) For the random variable $X_2$, compute $E[X_2]$ using the definition given in Eq. (3.8) on page 104 in the book. Show your work.

Solution

$$
\begin{align*}
E[X_2] &= \sum_{k=2}^{12} k p_{X_1}(k) \\
&= 2 \left( \frac{1}{36} \right) + 3 \left( \frac{2}{36} \right) + 4 \left( \frac{3}{36} \right) + 5 \left( \frac{4}{36} \right) + 6 \left( \frac{5}{36} \right) + 7 \left( \frac{6}{36} \right) \\
&\quad + 8 \left( \frac{5}{36} \right) + 9 \left( \frac{4}{36} \right) + 10 \left( \frac{3}{36} \right) + 11 \left( \frac{2}{36} \right) + 12 \left( \frac{1}{36} \right) \\
&= \frac{(2+12)1 + (3+11)2 + (4+10)3 + (5+9)4 + (6+8)5 + (7)6}{36} \\
&= \frac{14 \cdot 1 + 14 \cdot 2 + 14 \cdot 3 + 14 \cdot 4 + 14 \cdot 5 + 14 \cdot 6}{36} \\
&= \frac{14 \cdot (1+2+3+4+5+6) + 14 \cdot 7}{36} = \frac{14 \cdot 21 + 14 \cdot 7}{36} = \frac{252}{36} = 7.
\end{align*}
$$