1. (26 points) For Problem 2 in HW #6, we defined a random variable (on the sample space defined in problem 1 of HW#6) by $X_2((i, j)) = i + j$. We then calculated the probability mass function (PMF) and computed $E[X_2].$

1.a. (4 points) Let $B = \{(i, j) \mid i = j\}$. Calculate $p_{X_2}(x \mid B)$, the conditional PMF of $X_2$ given $B$, for each $x \in \mathbb{R}$. Clearly justify your answer using the definition of conditional PMF.

Solution

From the solution to 1.a in HW#6, we have

$$P[B] = P[(i, j) \mid i = j] = \sum_{i=j} P[(i, j)] = 6 \times \frac{1}{36} = \frac{1}{6}.$$ 

From the definition of conditional PMF, we have

$$p_{X_2}(x \mid B) = \frac{P[X = x \mid B]}{P[B]} = \frac{P[(i, j) \mid i + j = x] \cap \{(i, j) \mid i = j\}}{P[(i, j) \mid i = j]}.$$ 

Looking carefully at the set in the numerator, we see that

$$\{(i, j) \mid i + j = x\} \cap \{(i, j) \mid i = j\} = \begin{cases} \left\{\left(\frac{x}{2}, \frac{x}{2}\right)\right\} & \text{for } x = 2, 4, 6, 8, 10, 12 \\ \emptyset & \text{otherwise} \end{cases}$$

And like any other pair $(i, j)$, $P\left[\left\{\left(\frac{x}{2}, \frac{x}{2}\right)\right\}\right] = \frac{1}{36}$ for $x = 2, 4, 6, 8, 10, 12$.

So the conditional PMF of $X_2$ given $B$ is

$$p_{X_2}(x \mid B) = \begin{cases} \frac{1}{36} = \frac{1}{6} & \text{for } x = 2, 4, 6, 8, 10, 12. \\ 0 & \text{all other } x \in \mathbb{R}. \end{cases}$$
1.b. (4 points) Now calculate \( m_{X_2|B} = E[X_2|B] \) the conditional expectation of \( X_2 \) given \( B \). Clearly justify your answer using the definition of conditional expectation.

\[ m_{X_2|B} = E[X_2|B] = \sum_{k=1}^{\infty} x_k p_{X_2}(x_k | B) = \sum_{k=1}^{6} (2k) p_{X_2}(2k|B) \]
\[ = \sum_{k=1}^{6} (2k) \frac{1}{6} = \frac{1 + 2 + 3 + 4 + 5 + 6}{3} = \frac{21}{3} = 7. \]

1.c. (4 points) Now calculate \( E[X_2^2|B] \) the conditional 2nd moment of \( X_2 \) given \( B \). Clearly justify your answer using the definition of conditional expectation.

\[ E[X_2^2|B] = \sum_{k=1}^{\infty} x_k^2 p_{X_2}(x_k | B) = \sum_{k=1}^{6} (2k)^2 p_{X_2}(2k|B) \]
\[ = \sum_{k=1}^{6} (2k)^2 \frac{1}{6} = \frac{2}{3} \sum_{k=1}^{6} (k)^2 = \frac{2(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)}{3} = \frac{182}{3} = \frac{602}{3}. \]

1.d. (6 points) Now, calculate \( p_{X_2}(x|B^c) \), the conditional PMF of \( X_2 \) given \( B^c \), the complement of \( B \), for each \( x \in \mathbb{R} \). Clearly justify your answer using the definition of conditional PMF and double-check your answer by showing that \( \sum_{k=1}^{\infty} p_{X_2}(x_k | B^c) = 1 \).

Solution

From the solution to 1.a above, we can calculate
\[ P[B^c] = 1 - P[B] = 1 - \frac{1}{6} = \frac{5}{6}. \]

From the definition of conditional PMF, we have
\[ p_{X_k}(x_k | B^c) = P \left[ X = x_k | B^c \right] = P \left[ A_k | B^c \right] = \frac{P \left[ A_k \cap B^c \right]}{P[B^c]} \]

where for \( k = 2, 3, \ldots, 12, \ A_k = \{(i, j) \in S \mid X_2((i, j)) = i + j = k\} \).

Using this and the results of problem 2.c of HW#6, we have that the conditional PMF of \( X_2 \) given \( B^c \) is

\[ p_{X_2}(2 | B^c) = P \left[ A_2 | B^c \right] = P \left[ \{(1,1)\} | B^c \right] = \frac{P \left[ \{(1,1)\} \cap B^c \right]}{P[B^c]} = \frac{P[\emptyset]}{\frac{5}{6}} = 0, \]

\[ p_{X_2}(3 | B^c) = P \left[ A_3 | B^c \right] = P \left[ \{(1,2),(2,1)\} | B^c \right] = \frac{P \left[ \{(1,2),(2,1)\} \cap B^c \right]}{P[B^c]} = \frac{\frac{5}{6}}{\frac{5}{6}} = \frac{1}{15}, \]

\[ p_{X_2}(4 | B^c) = P \left[ A_4 | B^c \right] = P \left[ \{(1,3),(3,1),(2,2)\} | B^c \right] = \frac{P \left[ \{(1,3),(3,1),(2,2)\} \cap B^c \right]}{P[B^c]} = \frac{\frac{5}{6}}{\frac{5}{6}} = \frac{1}{15}. \]

Similarly, we can calculate

\[ p_{X_2}(5 | B^c) = P \left[ A_5 | B^c \right] = P \left[ \{(1,4),(4,1),(2,3),(3,2)\} | B^c \right] = \frac{\frac{4}{5}}{\frac{5}{6}} = \frac{2}{15}, \]

\[ p_{X_2}(6 | B^c) = P \left[ A_6 | B^c \right] = P \left[ \{(1,5),(5,1),(2,4),(4,2),(3,3)\} | B^c \right] = \frac{\frac{4}{5}}{\frac{5}{6}} = \frac{2}{15}, \]

\[ p_{X_2}(7 | B^c) = P \left[ A_7 | B^c \right] = P \left[ \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\} | B^c \right] = \frac{\frac{4}{5}}{\frac{5}{6}} = \frac{3}{15}, \]

\[ p_{X_2}(8 | B^c) = P \left[ A_8 | B^c \right] = P \left[ \{(2,6),(6,2),(3,5),(5,3),(4,4)\} | B^c \right] = \frac{\frac{4}{5}}{\frac{5}{6}} = \frac{2}{15}, \]

\[ p_{X_2}(9 | B^c) = P \left[ A_9 | B^c \right] = P \left[ \{(3,6),(6,3),(4,5),(5,4)\} | B^c \right] = \frac{\frac{4}{5}}{\frac{5}{6}} = \frac{2}{15}, \]

\[ p_{X_2}(10 | B^c) = P \left[ A_{10} | B^c \right] = P \left[ \{(4,6),(6,4),(5,5)\} | B^c \right] = \frac{\frac{3}{5}}{\frac{5}{6}} = \frac{1}{15}, \]

\[ p_{X_2}(11 | B^c) = P \left[ A_{11} | B^c \right] = P \left[ \{(5,6),(6,5)\} | B^c \right] = \frac{\frac{3}{5}}{\frac{5}{6}} = \frac{1}{15}, \]

\[ p_{X_2}(12 | B^c) = P \left[ A_{12} | B^c \right] = P \left[ \{(6,6)\} | B^c \right] = 0. \]

Note that \( \sum_{k=1}^{12} p_{X_2}(x_k | B^c) = \frac{0 + 1 + 1 + 2 + 2 + 3 + 2 + 2 + 1 + 1 + 0}{15} = 1. \).
1.e. (4 points) Now calculate $E[X_2^2|B^c]$ the conditional 2nd moment of $X_2$ given $B^c$, the complement of $B$. Clearly justify your answer using the definition of conditional expectation.

Solution

From the definition of conditional expectation, we have

$$E[X_2^2|B^c] = \sum_{k=2}^\infty k^2 p_{X_2}(k|B^c)$$

$$= 2^2 + 3^2 \left( \frac{1}{15} \right) + 4^2 \left( \frac{1}{15} \right) + 5^2 \left( \frac{1}{15} \right) + 6^2 \left( \frac{1}{15} \right) + 7^2 \left( \frac{1}{15} \right)$$

$$+ 8^2 \left( \frac{1}{15} \right) + 9^2 \left( \frac{1}{15} \right) + 10^2 \left( \frac{1}{15} \right) + 11^2 \left( \frac{1}{15} \right) + 12^2 (0)$$

$$= \frac{9+16+50+72+147+128+162+100+121}{15}$$

$$= \frac{805}{15} = 53\frac{2}{3}.$$

1.f. (4 points) In HW #7, we showed that $E[X_2^2] = \frac{1974}{36} = \frac{329}{6} = 54.83$. Using your results from 1.c and 1.e above, show that the Total Expectation Theorem holds in this case for $X_2$, the partition $\{B, B^c\}$, and $g(x) = x^2$.

Solution

The Total Expectation Theorem says

$$E[g(X)] = \sum_{i=1}^n E[g(X)|B_i] P[B_i].$$

Using our results from 1.c and 1.e above, we have

$$\sum_{i=1}^n E[X_2^2|B_i] P[B_i] = E[X_2^2|B] P[B] + E[X_2^2|B^c] P[B^c]$$

$$= \left( \frac{182}{3} \right) \frac{1}{6} + \left( \frac{805}{15} \right) \frac{5}{6} = \frac{987}{18} = \frac{329}{6} = 54.83 = E[X_2^2].$$