1. (8 points) A single fair 4-sided die is tossed once. Let the random variable $X_1 = \text{the number of dots on the face of the die}$. 

1.a. (4 points) Write out the PMF $p_{X_1}(k)$ and the CDF $F_{X_1}(x)$ of this RV. Graph the CDF.

**Solution**

Since it’s a fair 4-sided die, each side has probability $1/4$, so $p_{X_1}(x_k) = 0.25$ for $x_k = 1, 2, 3, 4$. Hence

$$F_{X_1}(x) = \begin{cases} 
0 & x < 1 \\
p_{X_1}(1) = 0.25 & 1 \leq x < 2 \\
p_{X_1}(1) + p_{X_1}(2) = 0.5 & 2 \leq x < 3 \\
p_{X_1}(1) + p_{X_1}(2) + p_{X_1}(3) = 0.75 & 3 \leq x < 4 \\
p_{X_1}(1) + p_{X_1}(2) + p_{X_1}(3) + p_{X_1}(4) = 1 & 4 \leq x
\end{cases}$$

![Graph of CDF](image1)

1.b. (4 points) Calculate and show on your graph $F_{X_1}(0), F_{X_1}(1.5), F_{X_1}(4^-), F_{X_1}(4.25)$.

**Solution**

$$F_{X_1}(0) = 0, F_{X_1}(1.5) = 0.25, F_{X_1}(4^-) = 0.75, F_{X_1}(4.25) = 1.$$
2. (8 points) A single fair 4-sided die is tossed once. Define a new random variable $X_2 = \frac{1}{\text{the number of dots on the face of the die}}$.

2.a. (3 points) What is $S_{x_2}$ in this case? Carefully write out the PMF $p_{x_2}(k)$ of this RV.

**Solution**

$S_{x_2} = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$.

Since it’s a fair 4-sided die, each side has probability $\frac{1}{4}$, so $p_{x_2}(x_k) = 0.25$ for $x_k = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$.

2.b. (5 points) Write out the CDF $F_{x_2}(x)$ of this RV and graph it.

**Solution**

$$F_{x_2}(x) = \begin{cases} 0 & x < \frac{1}{4} \\ p_{x_2}(\frac{1}{4}) = 0.25 & \frac{1}{4} \leq x < \frac{1}{3} \\ p_{x_2}(\frac{1}{4}) + p_{x_2}(\frac{1}{3}) = 0.5 & \frac{1}{3} \leq x < \frac{1}{2} \\ p_{x_2}(\frac{1}{4}) + p_{x_2}(\frac{1}{3}) + p_{x_2}(\frac{1}{2}) = 0.75 & \frac{1}{2} \leq x < 1 \\ p_{x_2}(\frac{1}{4}) + p_{x_2}(\frac{1}{3}) + p_{x_2}(\frac{1}{2}) + p_{x_2}(1) = 1 & 1 \leq x \end{cases}$$
3. (12 points) A random variable $X$ has CDF given by

$$F_X(x) = \begin{cases} 
0 & x < -3 \\
c(x + 3)^2 & -3 \leq x < 2 \\
1 & 2 \leq x
\end{cases}$$

3.a. (4 points) Find the constant $c$ which makes $X$ a continuous random variable. Justify your answer.

**Solution**

$X$ is continuous if and only if $F_X(x)$ is continuous, which happens when $c(x + 3)^2 = 1$ at $x = 2$. ($F_X(x)$ is already continuous everywhere else.) That is, when $c(2 + 3)^2 = 1$, i.e. when $c = \frac{1}{37} = 0.04$. So $X$ is continuous if and only if

$$F_X(x) = \begin{cases} 
0 & x < -3 \\
0.04(x + 3)^2 & -3 \leq x < 2 \\
1 & 2 \leq x
\end{cases}$$

3.b. (4 points) Carefully compute $P[X > 1]$ and $P[-2 < X \leq 0]$ for the correct value of $c$.

**Solution**

$$P[X > 1] = 1 - P[X \leq 1] = 1 - F_X(1) = 1 - 0.04(1 + 3)^2 = 1 - 0.64 = 0.36.$$  

$$P[-2 < X \leq 0] = F_X(0) - F_X(-2) = 0.04(0 + 3)^2 - 0.04(-2 + 3)^2 = 0.04(9 - 1) = 0.32.$$
3.c. (4 points) Carefully compute the value of $a$ for which $P[X > a] = 0.4$, for the correct value of $c$.

Solution

$$P[X > a] = 1 - P[X \leq a] = 1 - F_X(a) = 1 - 0.04(a + 3)^2$$

$$= 1 - 0.04(a^2 + 6a + 9) = -0.04a^2 - 0.24a + 1 - 0.36$$

$$= -0.04a^2 - 0.24a + 0.64.$$

So $0.4 = P[X > a] = -0.04a^2 - 0.24a + 0.64$ if and only if

$$0 = -0.04a^2 - 0.24a + 0.24$$

iff $0 = a^2 + 6a - 6$

iff $a = \frac{-6 \pm \sqrt{36 + 24}}{2} = -3 \pm \frac{\sqrt{60}}{2} = -3 \pm 3.873.$

Since we can see that for $a = -3 - 3.873 = -6.873$, we have $a < -3$, so $P[X > a] = 1 - F_X(x) = 1 - 0 = 1$.

Thus, the correct value of $a$ must be $a = -3 + 3.873 = 0.873$.

Checking, we see that

$$P[X > 0.873] = 1 - F_X(0.873) = 1 - 0.04(0.873 + 3)^2 = 1 - 0.04(15) = 0.4.$$