1. (4 points) Let $X$ and $Y$ be discrete RV's with joint PMF $p_{X,Y}(x,y)$ and marginal PMF's $p_X(x)$ and $p_Y(y)$ as given in the following table:

<table>
<thead>
<tr>
<th>$p_{X,Y}(x,y)$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$p_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.06</td>
<td>0.04</td>
<td>0.35</td>
<td>0.45</td>
</tr>
<tr>
<td>0</td>
<td>0.14</td>
<td>0.05</td>
<td>0.21</td>
<td>0.40</td>
</tr>
<tr>
<td>$-1$</td>
<td>0.10</td>
<td>0.03</td>
<td>0.02</td>
<td>0.15</td>
</tr>
<tr>
<td>$p_Y(y)$</td>
<td>0.30</td>
<td>0.12</td>
<td>0.58</td>
<td></td>
</tr>
</tbody>
</table>

Are $X$ and $Y$ independent? Justify your answer.

**Solution**

No, $X$ and $Y$ are not independent. In fact, $p_{X,Y}(x_j,y_k) \neq p_X(x_j)p_Y(y_k)$ for each of the nine pairs $(x_j,y_k)$ with $x_j = -1$, $0$, or $1$, and $y_k = -1$, $0$, or $1$. Any one of these proves that $X$ and $Y$ are not independent.

2. (4 points) Let $X$ and $Y$ be discrete RV's with joint PMF $p_{X,Y}(x,y)$ and marginal PMF's $p_X(x)$ and $p_Y(y)$ as given in the following table:

<table>
<thead>
<tr>
<th>$p_{X,Y}(x,y)$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$p_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.15</td>
<td>0.05</td>
<td>0.30</td>
<td>0.50</td>
</tr>
<tr>
<td>0</td>
<td>0.12</td>
<td>0.04</td>
<td>0.24</td>
<td>0.40</td>
</tr>
<tr>
<td>$-1$</td>
<td>0.03</td>
<td>0.01</td>
<td>0.06</td>
<td>0.10</td>
</tr>
<tr>
<td>$p_Y(y)$</td>
<td>0.30</td>
<td>0.10</td>
<td>0.60</td>
<td></td>
</tr>
</tbody>
</table>

Are $X$ and $Y$ independent? Justify your answer.

**Solution**

Yes, $X$ and $Y$ are independent. Checking each of the nine pairs $(x_j,y_k)$ with $x_j = -1$, $0$, or $1$, and $y_k = -1$, $0$, or $1$, we see that $p_{X,Y}(x_j,y_k) = p_X(x_j)p_Y(y_k)$ in each case:

- $0.15 = 0.3 \times 0.5$
- $0.05 = 0.1 \times 0.5$
- $0.30 = 0.6 \times 0.5$
- $0.12 = 0.3 \times 0.4$
- $0.04 = 0.1 \times 0.4$
- $0.24 = 0.6 \times 0.4$
- $0.03 = 0.3 \times 0.1$
- $0.01 = 0.1 \times 0.1$
- $0.06 = 0.6 \times 0.1$
3. (8 points) Let $X$ and $Y$ be jointly Gaussian RV’s, where $X$ has mean $\mu_X$ and variance $\sigma_X^2$, where $Y$ has mean $\mu_Y$ and variance $\sigma_Y^2$, and $\rho = 0$.

3.a. (4 points) Write the joint CDF of $X$ and $Y$ in terms of $\mu_X$, $\sigma_X^2$, $\mu_Y$, $\sigma_Y^2$, and the $Q$ function.

Solution

Since $\rho = 0$, we have from the slides and book that $X$ and $Y$ are independent. Thus,

$$F_{XY}(x,y) = F_X(x)F_Y(y).$$

Since $X$ has mean $\mu_X$ and variance $\sigma_X^2$, we have from Eq. (4.49) in the book that

$$F_X(x) = \Phi \left( \frac{x - \mu_X}{\sigma_X} \right) = 1 - Q \left( \frac{x - \mu_X}{\sigma_X} \right).$$

Similarly, since $Y$ has mean $\mu_Y$ and variance $\sigma_Y^2$, we have

$$F_Y(y) = \Phi \left( \frac{y - \mu_Y}{\sigma_Y} \right) = 1 - Q \left( \frac{y - \mu_Y}{\sigma_Y} \right).$$

Putting these 3 equations together, we get

$$F_{XY}(x,y) = F_X(x)F_Y(y) = \left[ 1 - Q \left( \frac{x - \mu_X}{\sigma_X} \right) \right] \left[ 1 - Q \left( \frac{y - \mu_Y}{\sigma_Y} \right) \right].$$
3.b. (4 points, 2 points each) Use your formula in part a to compute the following values:

3.b.i. \( F_{XY}(0,0) \) when \( X \) and \( Y \) are each Standard Gaussian.

Solution

\[
F_{XY}(x,y) = \left[ 1 - Q\left( \frac{x-\mu_X}{\sigma_X} \right) \right] \left[ 1 - Q\left( \frac{y-\mu_Y}{\sigma_Y} \right) \right] \\
= \left[ 1 - Q\left( \frac{0-0}{1} \right) \right] \left[ 1 - Q\left( \frac{0-0}{1} \right) \right] \\
= \left[ \frac{1}{2} \right] \left[ \frac{1}{2} \right] \\
= \frac{1}{4}.
\]

This makes sense because \( F_{XY}(0,0) \) is the likelihood that \( X \leq 0 \) and \( Y \leq 0 \), which is the quarter plane and \( f_{XY}(x,y) \) is symmetric in every direction, so the probability of each quarter plane must be the same.

3.b.ii. \( F_{XY}(0,0) \) when \( \mu_X = -1, \sigma_X^2 = 1, \mu_Y = 1, \sigma_Y^2 = 4. \)

Solution

\[
F_{XY}(x,y) = \left[ 1 - Q\left( \frac{x-\mu_X}{\sigma_X} \right) \right] \left[ 1 - Q\left( \frac{y-\mu_Y}{\sigma_Y} \right) \right] \\
= \left[ 1 - Q\left( \frac{0-(-1)}{1} \right) \right] \left[ 1 - Q\left( \frac{0-1}{2} \right) \right] \\
= \left[ 1 - Q(1) \right]\left[ 1 - Q(-0.5) \right] \\
= \left[ 1 - 0.159 \right]\left[ Q(0.5) \right] \\
= \left[ 0.841 \right]\left[ 0.309 \right] \\
= 0.260.
\]