1. (16 points) Consider two discrete random variables $X$ and $Y$ with joint PMF

$$p_{XY}(j,k) = \begin{cases} \frac{C(|j|+|k|)}{12} & \text{for } j = -1, 0, 1; \ k = -1, 0, 1. \\ 0 & \text{elsewhere} \end{cases}$$

1.a. (4 points) Draw a picture of $p_{XY}(j,k)$, and then compute the value of $C$, which makes $p_{XY}(j,k)$ a proper joint PMF.

**Solution**

First, a picture:

We have

$$1 = \sum_{k=-1}^{1} \sum_{j=-1}^{1} p_{XY}(j,k) = 4 \times C + 4 \times 2C = 12C.$$ So $C = \frac{1}{12}$.

Thus,

$$p_{XY}(j,k) = \begin{cases} \frac{1}{12}(|j|+|k|) & \text{for } j = -1, 0, 1; \ k = -1, 0, 1. \\ 0 & \text{elsewhere} \end{cases}$$

And
1.b. (4 points) Compute the marginal PMF of $X$, $p_X(j)$ and the marginal PMF of $Y$, $p_Y(k)$.

Solution

Summing down the columns, we have

$$p_X(j) = \sum_{k=-1}^{1} p_{XY}(j,k) = \begin{cases} \frac{3}{12} & \text{for } j = 1 \\ \frac{2}{12} & \text{for } j = 0 \\ \frac{5}{12} & \text{for } j = -1 \\ 0 & \text{elsewhere} \end{cases}$$

Summing across the rows, we have

$$p_Y(k) = \sum_{j=-1}^{1} p_{XY}(j,k) = \begin{cases} \frac{3}{12} & \text{for } k = 1 \\ \frac{2}{12} & \text{for } k = 0 \\ \frac{5}{12} & \text{for } k = -1 \\ 0 & \text{elsewhere} \end{cases}$$


Solution

$$E[X] = E[Y] = -1(\frac{3}{12}) + 0(\frac{2}{12}) + 1(\frac{5}{12}) = 0$$

$$E[XY] = -1(1)(\frac{3}{12}) + 0(1)(\frac{2}{12}) + 1(1)(\frac{5}{12})$$

$$= -1(0)(\frac{1}{12}) + 0(0)(\frac{1}{12}) + 1(0)(\frac{1}{12})$$

$$= 0$$

$$= 0$$

$$= 0$$

So $E[XY] = 0 = E[X]E[Y]$. Hence $X$ and $Y$ are uncorrelated.
1.d. (4 points) Are $X$ and $Y$ independent? Justify your answer.

Solution

No, $X$ and $Y$ are not independent. In fact, $p_{XY}(j,k) \neq p_X(j)p_Y(k)$ for each of the nine pairs $(j,k)$ with $j=-1$, 0, or 1, and $k=-1$, 0, or 1. Any one of these proves that $X$ and $Y$ are not independent.

\[
\frac{2}{12} \neq \frac{5}{12} \left( \frac{5}{12} \right) \quad \frac{1}{12} \neq \frac{2}{12} \left( \frac{5}{12} \right) \quad \frac{2}{12} \neq \frac{5}{12} \left( \frac{5}{12} \right)
\]

\[
\frac{1}{12} \neq \frac{5}{12} \left( \frac{5}{12} \right) \quad 0 \neq \frac{2}{12} \left( \frac{5}{12} \right) \quad \frac{1}{12} \neq \frac{2}{12} \left( \frac{5}{12} \right)
\]

\[
\frac{2}{12} \neq \frac{5}{12} \left( \frac{5}{12} \right) \quad \frac{1}{12} \neq \frac{2}{12} \left( \frac{5}{12} \right) \quad \frac{2}{12} \neq \frac{5}{12} \left( \frac{5}{12} \right)
\]

2. (16 points) Let $X$ and $Y$ be two random variables with joint PDF

\[
f_{XY}(x,y) = \begin{cases} C & 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}
\]

2.a. (4 points) Draw a picture of the domain of $f_{XY}(x,y)$ and find the constant $C$ that makes $f_{XY}(x,y)$ a proper joint density function for $X$ and $Y$. Justify your answer.

Solution

The area of the triangle = $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

Since we must have

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \, dx \, dy = C \times \text{(area of triangle)} = 1,
\]

then we must have

\[
f_{XY}(x,y) = 2 \quad \text{for} \ 0 \leq y \leq x \leq 1
\]

and zero elsewhere.
2.b. (4 points) Compute the marginal PDF’s of $X$ and $Y$. Show your work.

Solution

\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_{0}^{x} 2 \, dy = [2y]_{0}^{x} = 2x - 0 = 2x \quad \text{for} \quad -1 \leq x \leq 1
\]

and zero elsewhere.

Similarly, integrating along the blue line in the figure below, we have

\[
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx = \int_{y}^{1} 2 \, dx = [2x]_{y}^{1} = 2 - 2y = 2 - 2(1 - y) \quad \text{for} \quad -1 \leq y \leq 1
\]

and zero elsewhere.

[Diagram of a triangular region with axes labeled $x$ and $y$, and a line at $y = 2$.]

Solution

$$E[X] = \int_{-\infty}^{\infty} xf_X(x) \, dx = \int_{0}^{1} 2x^2 \, dx = \left[ \frac{2x^3}{3} \right]_{0}^{1} = \frac{2}{3} - 0 = \frac{2}{3},$$

$$E[Y] = \int_{-\infty}^{\infty} yf_Y(y) \, dy = \int_{0}^{1} (2y - 2y^3) \, dy = \left[ y^2 - \frac{2y^4}{3} \right]_{0}^{1} = 1 - \frac{2}{3} - 0 = \frac{1}{3},$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{X,Y}(x,y) \, dxdy = \int_{0}^{1} \int_{0}^{1} xy^2 \, dxdy = \int_{0}^{1} \left[ \frac{1}{2}x^2 \right]_{0}^{1} \, dy = \int_{0}^{1} \left[ y^3 \right]_{0}^{1} \, dy = \frac{1}{2} - \frac{1}{4} - 0 = \frac{1}{4}.$$

Alternatively,

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{X,Y}(x,y) \, dxdy = \int_{0}^{1} \int_{0}^{1} 2xy \, dxdy = \int_{0}^{1} \left[ \frac{1}{2}y^2 \right]_{0}^{1} \, dx = \int_{0}^{1} \left[ x^2 \right]_{0}^{1} \, dx = \frac{1}{4} - 0 = \frac{1}{4}.$$

Either way, we have $\frac{1}{4} \neq -\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$. So $X$ and $Y$ are correlated.

2.d. (4 points) Are $X$, $Y$ independent? Justify your answer.

Solution

No. Since we showed in part c that $X$ and $Y$ are correlated, they cannot be independent. Because if they were independent, they would be uncorrelated.