CDFs & PDFs of a Function of a RV

\[ Y = g(X) \]

Section 4.5
Expectation of a Function of a RV

- Last time, we saw how to compute the expected value of $Y$ where $Y$ is a function of a RV $X$, that is, $Y = g(X)$, via

\[ E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx \]

- Today, we will find ways of computing the CDF and PDF of $Y$.

- We will work case by case. There is no universal method for doing this.

- Our first approach is to try to directly evaluate $F_Y(y)$. 
**Example from Last time**

- $X$ uniform on $[0,1]$: $f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$
- $Y = g(X) = e^X$
- $X$ takes values in $[0,1]$, so $Y$ takes values in $[1,e]$.
- By definition, the CDF of $Y$ is
  \[ F_Y(y) = P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] \]
  \[ = \int_0^{\ln y} f_X(x) \, dx = \int_0^{\ln y} 1 \, dx = \ln y \quad \text{for } y \in [1,e] \]
- Hence $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \ln y = \frac{1}{y}$ for $y \in [1,e]$. 

Range is very important.
Ex 2: \( g(X) = aX + b \)

- \( g(x) = aX + b \) is called an “affine” function.
- Affine functions include linear functions, \( g(x) = aX \), as a special case.
- Assume \( a > 0 \), then

\[
F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = P\left[X \leq \left(\frac{y-b}{a}\right)\right] = F_X\left(\frac{y-b}{a}\right)
\]

- So \( f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \)

by the chain rule.
Ex 2: \( g(X) = aX + b \) continued

- Assume \( a < 0 \), then

\[
F_Y(y) = P[Y \leq y] = P[aX + b \leq y]
\]

\[
= P\left[X \geq \left(\frac{y-b}{a}\right)\right] = 1 - F_X\left(\frac{y-b}{a}\right)
\]

- So \( f_Y(y) = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right] = \frac{1}{-a} f_X\left(\frac{y-b}{a}\right) \)

by the chain rule.
Ex 2: \( g(X) = aX + b \) continued

- We just showed that for any \( a > 0 \),
  \[ f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \]

- And for any \( a < 0 \),
  \[ f_Y(y) = \frac{1}{-a} f_X\left(\frac{y-b}{a}\right) \]

- So for any \( a \neq 0 \), we have
  \[ f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \]
\[ g(X) = aX + b \text{ for } X \text{ Gaussian} \]

- The Gaussian PDF is
  \[
  f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} \exp\left[ -\frac{(x-\mu)^2}{2\sigma^2} \right]
  \]

- So, if \( Y = aX + b \), then
  \[
  f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi |a| \sigma}} \exp\left[ -\frac{(y-(a\mu+b))^2}{2(a\sigma)^2} \right]
  \]

- So \( Y = aX + b \) is Gaussian with mean \((a\mu + b)\) and variance \((a\sigma)^2\) and standard deviation \(|a|\sigma\).

- An affine function of a Gaussian RV is Gaussian!

- A linear function of a Gaussian RV is Gaussian!
The Effect of Limiters

- Often in processing data, we put limits on values.
- For example, we may use a soft limiter
  \[ g(x) = \begin{cases} 
  x_1 & \text{for } x < x_1 \\
  x & \text{for } x_1 \leq x \leq x_2 \\
  x_2 & \text{for } x > x_2 
  \end{cases} \]
- Or a blanker
  \[ b(x) = \begin{cases} 
  0 & \text{for } x < x_1 \\
  x & \text{for } x_1 \leq x \leq x_2 \\
  0 & \text{for } x > x_2 
  \end{cases} \]
- A blanker in your TV protects against lightning.
Ex 3: Soft Limiter

- A soft limiter $Y = g(X)$

$$g(x) = \begin{cases} 
    x_1 & \text{for } x < x_1 \\
    x & \text{for } x_1 \leq x \leq x_2 \\
    x_2 & \text{for } x > x_2 
\end{cases}$$

$$F_Y(y) = P[Y \leq y] = \begin{cases} 
    0 & \text{for } y < x_1 \\
    F_X(y) & \text{for } x_1 \leq y < x_2 \\
    1 & \text{for } x_2 \leq y 
\end{cases}$$
Ex 3: Soft Limiter continued

- A soft limiter $Y = g(X)$ yields

\[
F_Y(y) = \begin{cases} 
0 & \text{for } y < x_1 \\
F_X(y) & \text{for } x_1 \leq y < x_2 \\
1 & \text{for } x_2 \leq y 
\end{cases}
\]

\[
f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 
0 & \text{for } y < x_1 \\
F_X(x_1) \delta(y - x_1) & \text{for } y = x_1 \\
f_X(x) & \text{for } x_1 < y < x_2 \\
[1 - F_X(x_2)] \delta(y - x_2) & \text{for } y = x_2 \\
0 & \text{for } x_2 < y 
\end{cases}
\]
**Ex 3: Soft Limiter continued**

- A soft limiter \( Y = g(X) \) yields

\[
f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 
0 & \text{for } y < x_1 \\
F_X(x_1)\delta(y-x_1) & \text{for } y = x_1 \\
f_X(x) & \text{for } x_1 < y < x_2 \\
[1-F_X(x_2)]\delta(y-x_2) & \text{for } y = x_2 \\
0 & \text{for } x_2 < y 
\end{cases}
\]

An example of a “mixed” PDF
Ex 4: Hard Limiter

- A hard limiter $Y = g(X)$
  
  $g(x) = \begin{cases} 
  -1 & \text{for } x \leq 0 \\
  1 & \text{for } x > 0 
  \end{cases}$

- $Y$ only takes 2 values: +1 and −1, so $Y$ is a discrete RV.

- So we have $P[Y = -1] = P[X \leq 0] = F_X(0)$
  
  and $P[Y = 1] = P[X > 0] = 1 - F_X(0)$.

- Thus
  
  $F_Y(y) = \begin{cases} 
  0 & y < -1 \\
  F_X(0) & -1 \leq y < 1 \\
  1 & y \geq 1 
  \end{cases}$
Ex 4: Hard Limiter continued

- A hard limiter $Y = g(X)$ yields

$$F_Y(y) = \begin{cases} 
0 & y < -1 \\
F_X(0) & -1 \leq y < 1 \\
1 & y \geq 1 
\end{cases}$$
**Ex 4: Hard Limiter** continued

- A hard limiter $Y = g(X)$ yields $F_Y(y) = \begin{cases} 
0 & y < -1 \\
F_X(0) & -1 \leq y < 1 \\
1 & y \geq 1 
\end{cases}$

- Thus $f_Y(y) = \frac{dF_Y(y)}{dy} = F_X(0) \delta(y + 1) + [1 - F_X(0)] \delta(y - 1)$.

A discrete PDF
Ex 4: Hard Limiter continued

- Now assume that $X$ is Gaussian with mean 0 and variance 4.

- Then $F_X(0) = \Phi\left(\frac{0}{2}\right) = \Phi(0) = 0.5$.

- And $f_Y(y) = 0.5\delta(y+1) + 0.5\delta(y-1)$.

A discrete PDF
Ex 5: \( g(x) = |x| \)

- Since \( y = g(x) = |x| \), we have for \( y > 0 \),

\[
F_Y(y) = P[Y \leq y] = P[-y \leq X \leq y]
\]

- For continuous \( F_X \), this means for \( y > 0 \)

\[
F_Y(y) = P[-y \leq X \leq y] = F_X(y) - F_X(-y)
\]

- And

\[
f_Y(y) = \frac{d}{dy}[F_X(y) - F_X(-y)] = f_X(y) + f_X(-y) \quad \text{for} \quad y > 0.
\]
Ex 6: Positive Part of Gaussian

- Let $X$ be a Gaussian random variable with mean 2 and variance 4. The reward in a system is given by $Y = (X)^+$ where $y = (x)^+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$

(1) So $F_Y(y) = P[Y \leq y] = \begin{cases} P[X \leq y] & y \geq 0 \\ 0 & y < 0 \end{cases}$

- Find the PDF of $Y$ by taking the derivative of (1) to get: $f_Y(y) = F_X(0)\delta(y) + f_X(y)u(y)$ where $u$ is the unit step function and

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-2)^2}{8}}$$
Ex 6: Positive Part of Gaussian continued

- We just showed \( f_Y(y) = F_X(0)\delta(y) + f_X(y)u(y) \)

- We just need to calculate \( F_X(0) \)

\[
F_X(0) = \Phi\left(\frac{0 - 2}{2}\right) = \Phi(-1) = 1 - Q(-1) = Q(1) = 0.159
\]

- So, \( f_Y(y) = 0.159\delta(y) + f_X(y)u(y) \)

where \( u \) is the unit step function.
Ex 7: Signal Processing

- The signal $X$ has CDF:

$$F_X(x) = \begin{cases} 
0 & x < -1 \\
0.5 & -1 \leq x < 0 \\
\frac{1}{2}(1 + x) & 0 \leq x < 1 \\
1 & 1 \leq x 
\end{cases}$$

- $X$ is amplified by 2 and shifted by 3. Let $Y = 2X + 3$.

- Find the CDF and PDF of $Y$.

$$F_Y(y) = P[Y \leq y] = P[2X + 3 \leq y] = P\left[X \leq \frac{1}{2}(y - 3)\right] = F_X\left(\frac{1}{2}(y - 3)\right)$$
Ex 7: Signal Processing continued

\[ F_Y(y) = F_X\left(\frac{1}{2}(y - 3)\right) = \begin{cases} 
0 & \frac{1}{2}(y - 3) < -1 \\
0.5 & -1 \leq \frac{1}{2}(y - 3) < 0 \\
\frac{1}{2}\left(1 + \frac{1}{2}(y - 3)\right) & 0 \leq \frac{1}{2}(y - 3) < 1 \\
1 & 1 \leq \frac{1}{2}(y - 3)
\end{cases} \]

\[ F_Y(y) = \begin{cases} 
0 & y < 1 \\
0.5 & 1 \leq y < 3 \\
\frac{1}{4}(y - 1) & 3 \leq y < 5 \\
1 & 5 \leq y 
\end{cases} \]

So the PDF is \( f_Y(y) = 0.5\delta(x-1) + \begin{cases} 
\frac{1}{4} & 3 \leq y < 5 \\
0 & \text{else}
\end{cases} \)
Ex 7: Signal Processing  continued

\[ F_Y(y) = \begin{cases} 
0 & y < 1 \\
0.5 & 1 \leq y < 3 \\
\frac{1}{4} (y - 1) & 3 \leq y < 5 \\
1 & 5 \leq y 
\end{cases} \]

\[ f_Y(y) = 0.5 \delta(x - 1) + \begin{cases} 
\frac{1}{4} & 3 \leq y < 5 \\
0 & \text{else} 
\end{cases} \]
A General Formula for the PDF of $g(x)$
General Formula for PDF of $g(x)$

- In the first 2 cases we considered, we had $g(x) = e^x$ and $g(x) = ax + b$.

- In both these cases, $g(x)$ is one to one, i.e. if $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$.

- For $y = g(x) = |x|$, there are 2 values of $x$ that map into each $y > 0$.

- Consider a case where there up to $n$ values of $x$ that map into some given $y$.

- So there are real values $x_1, x_2, \ldots, x_n$ such that $g(x_1) = g(x_2) = \cdots = g(x_n) = y$. 
General Formula for PDF of $g(x)$ cont’d

- Assume $g(x)$ has derivative $g'(x)$.
- Let $y$ be a given value of $Y$.
- If there are real values $x_1, x_2, \ldots, x_n$ such that $g(x_1) = g(x_2) = \cdots = g(x_n) = y$, then

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|}$$

- Aside: This result is related to the Inverse Function Theorem in Calculus.
Check on Example from Last time

- $X$ uniform on $[0,1]$: $f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

- Let $g(x) = e^x$, then for a fixed $y$, there is only one solution to $y = e^x$ for $x \in [0,1]$

- That solution is $x_1 = \ln y$ for $y \in [1,e]$.

- Also, we have $g'(x) = e^x$

- Hence,

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{1}{|e^{\ln y}|} = \frac{1}{y} \quad \text{for } y \in [1,e].$$
Check on $g(X) = aX + b$ for $a \neq 0$

- Here, $g(x) = ax + b$, so $g'(x) = a$.

- And $y = g(x) = ax + b$, so $x_1 = \frac{y-b}{a}$ for $a \neq 0$.

- So for $a \neq 0$, we have

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- which checks!
Consider \( g(x) = x^2 \)

- Here \( g'(x) = 2x \).
- And \( y = g(x) = x^2 \) has 2 solutions for \( y > 0 \):
  \[ x_1 = \sqrt{y} \text{ and } x_2 = -\sqrt{y} \]
- So for \( y > 0 \),
  \[
f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} = \frac{f_X(\sqrt{y})}{|g'(\sqrt{y})|} + \frac{f_X(-\sqrt{y})}{|g'(-\sqrt{y})|}
  \]
  \[
  = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{-2\sqrt{y}} = \frac{1}{2\sqrt{y}} \left[ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right]
  \]
\[
g(x) = x^2 \quad \text{continued}
\]

- We just showed that for \( y > 0 \),
  \[
f_Y(y) = \frac{1}{2\sqrt{y}} \left[ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right]
\]
- If \( X \) is standard Gaussian, then \( f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \)
- So for \( y > 0 \),
  \[
f_Y(y) = \frac{1}{2\sqrt{y}} \left[ \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right] = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}
\]
$g(x) = \sin(x)$ and $X$ Uniform

- Let $X$ be uniform on $(-\pi, \pi)$,
  \[
f_X(x) = \begin{cases} 
  \frac{1}{2\pi} & \text{for } -\pi < x < \pi \\
  0 & \text{elsewhere}
\end{cases}
\]
- Let $g(x) = \sin(x)$ for $-\pi < x < \pi$.
- Then $g'(x) = \cos(x)$ for $-\pi < x < \pi$.
- And $y = g(x) = \sin(x)$ means $x = \pm \arcsin(y)$.
- $g'(x) = \cos(\pm \arcsin(y)) = \pm \sqrt{1-y^2}$
- So for $-1 < y < 1$, we have
  \[
f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} = \frac{1/2\pi}{\sqrt{1-y^2}} + \frac{1/2\pi}{\sqrt{1-y^2}} = \frac{1}{\pi \sqrt{1-y^2}}
\]