The Joint Prob Density Function (PDF) of Two Random Variables

Section 5.4
The Joint PDF of 2 RVs

- Recall: Two random variables $X$ and $Y$ on a prob space $(S, \mathcal{F}, P)$ have joint CDF defined by

\[ F_{X,Y}(x, y) = P[X \leq x, Y \leq y] \]

- The (joint) Prob Density Function (PDF) $f_{X,Y}(x,y)$ of $X$ and $Y$ is defined by

\[ f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} \]

- From the Fundamental Theorem of Calculus, we must have

\[ F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} \, d\hat{y} \]
Properties of the Joint PDF

1. \( f_{X,Y}(x, y) \geq 0 \) for all \((x, y) \in \mathbb{R}^2\).

2. \[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} d\hat{y} = F_{X,Y}(\infty, \infty) = 1.
\]

3. For any region \( B \) in the \( x\)-\( y \) plane, we have

\[
P[(X, Y) \in B] = \int_{B} \int f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} d\hat{y}
\]

4. Specifically, for \( B = \{a_1 < x \leq a_2, \ b_1 < y \leq b_2\} \)

\[
P[a_1 < X \leq a_2, \ b_1 < Y \leq b_2] = \int_{b_1}^{b_2} \int_{a_1}^{a_2} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} d\hat{y}
\]
Recall Ex: Uniform on Unit Square

- Here we want the prob of a set to be equal to its area.
- We showed:

\[
F_{XY}(x,y) = P[X \leq x, Y \leq y] = \begin{cases} 
0 & \text{for } x < 0 \text{ or } y < 0 \\
xy & \text{for } 0 \leq x, y \leq 1 \\
x & \text{for } 0 \leq x \leq 1, y > 1 \\
y & \text{for } 0 \leq y \leq 1, x > 1 \\
1 & \text{for } x > 1 \text{ and } y > 1 
\end{cases}
\]
**Ex: PDF of Uniform**

\[
F_{XY}(x, y) = P[X \leq x, Y \leq y] = \begin{cases} 
0 & x < 0 \text{ or } y < 0 \\
x & 0 \leq x \leq 1, \ y > 1 \\
y & 0 \leq y \leq 1, \ x > 1 \\
1 & x > 1 \text{ and } y > 1 
\end{cases}
\]

- So

\[
\frac{\partial F_{X,Y}(x, y)}{\partial x} = \begin{cases} 
0 & x < 0 \text{ or } y < 0 \\
y & 0 \leq x, \ y \leq 1 \\
1 & 0 \leq x \leq 1, \ y > 1 \\
0 & 0 \leq y \leq 1, \ x > 1 \\
0 & x > 1 \text{ and } y > 1 
\end{cases}
\]

- And the joint PDF is

\[
f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = \begin{cases} 
1 & 0 \leq x, y \leq 1 \\
0 & \text{elsewhere} 
\end{cases}
\]
Ex: PDF of Uniform

- The joint PDF is
  
  \[
  f_{X,Y}(x, y) = \begin{cases} 
  1 & 0 \leq x, y \leq 1 \\
  0 & \text{elsewhere}
  \end{cases}
  \]

- We know in general that
  
  \[
  P[(X, Y) \in B] = \iint_B f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x}d\hat{y}
  \]

- This is consistent with our goal for the Uniform:
  
  \[
  P[(X, Y) \in B] = \iint_B 1 \, d\hat{x}d\hat{y} = \text{area of } B
  \]

  for any set \( B \) inside the unit square (red area).
Marginal PDFs of $X$ and $Y$

- The PDF of $X$ can be found from the joint PDF by integrating out $y$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \quad \text{for each } x \in \mathbb{R}.$$  

- In this context, $f_X(x)$ is called the marginal PDF of $X$.

- And the marginal PDF of $Y$ obtained by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(\hat{x}, y) \, d\hat{x} \quad \text{for each } y \in \mathbb{R}.$$
Example: Marginal PDF of Uniform

- If $X$ and $Y$ are joint uniform, then the marginal PDF for $X$ is univariate uniform:

$$f_X(x) = \int_0^1 f_{X,Y}(x, \hat{y}) d\hat{y} = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

We integrated along this line for each $x$ to get $f_X(x)$.
Example: Two Exponential RVs

- Let \( F_{XY}(x, y) = \begin{cases} (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y}) & \text{for } x, y > 0 \\ 0 & \text{elsewhere} \end{cases} \)

- \( F_{XY}(x, y) \) is differentiable at every point \((x, y)\) with \(x > 0\) and \(y > 0\).

- So, \( \frac{\partial F_{XY}(x, y)}{\partial x} = \begin{cases} (\lambda_1 e^{-\lambda_1 x})(1 - e^{-\lambda_2 y}) & \text{for } x, y > 0 \\ 0 & \text{elsewhere} \end{cases} \)

- And \( f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = \begin{cases} \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 y} & \text{for } x, y > 0 \\ 0 & \text{elsewhere} \end{cases} \)
Jointly Gaussian RVs

Defined just like one Gaussian RV: in terms of its PDF
Two Jointly Gaussian RVs

- RVs $X$ and $Y$ are jointly Gaussian if
  \[
  f_{XY}(x,y) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}}\exp\left\{\left(-\frac{1}{2(1-\rho^2)}\right)[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}]\right\}
  \]

- The marginal distribution of $X$ is Gaussian with mean $\mu_1$ and variance $\sigma_1^2$.

- The marginal distribution of $Y$ is Gaussian with mean $\mu_2$ and variance $\sigma_2^2$.

- $\rho$ is the correlation coefficient which we will discuss in detail next time
  \[
  \rho = \frac{E[(X-\mu_1)(Y-\mu_2)]}{\sigma_1\sigma_2}
  \]
Example 1: Jointly Gaussian RVs

- Simplest case:
  \[ \mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1, \rho = 0 \]
  \[ f_{XY}(x, y) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[x^2 + y^2]\right\} \]
Example 2: Jointly Gaussian RVs

- Let $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0.4$

- Then $f_{XY}(x,y) = \frac{1}{2\pi(0.92)} \exp\left\{\frac{-[x^2 - 2(0.4)xy + y^2]}{2(0.84)}\right\}$
Example 3: Jointly Gaussian RVs

- Nice image of a Gaussian with $\sigma_1 = \sigma_2$, $\rho = 0$.
- Note the contours are circles: “Circularly symmetric”.

From: www.moserware.com
Example 4: Jointly Gaussian RVs

- Image of a Gaussian with $\rho$ closer to 1.
- Note that the contours are ovals along the line $x = y$. 

From: www.moserware.com
Another Example of a 2-RV PDF

- Let $f_{X,Y}(x, y) = \begin{cases} A(x + y) & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

- First compute $A$ via

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} A(x + y) \, dx \, dy$$

$$\frac{1}{A} = \int_{0}^{1} \left[ \frac{x^2}{2} + y \right]_{x=0}^{1} \, dy = \int_{0}^{1} \left[ \frac{1}{2} + y \right] \, dy = \left[ \frac{1}{2} + \frac{y^2}{2} \right]_{y=0}^{1} = 1$$

- So $A = 1$, and $f_{X,Y}(x, y) = \begin{cases} x + y & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$
Another Ex. of a 2-RV PDF continued

- \( f_{X,Y}(x, y) = \begin{cases} 
  x + y & 0 < x, y \leq 1 \\
  0 & \text{elsewhere} 
\end{cases} \)

- The marginal PDF of \( X \) for \( 0 < x \leq 1 \) is

\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, \hat{y}) \, d\hat{y} = \int_{0}^{1} [x + \hat{y}] \, d\hat{y}
\]

\[
= \left[ xy + \frac{y^2}{2} \right]_{y=0}^{1} = x + \frac{1}{2}
\]

zero elsewhere.

- By symmetry \( f_Y(y) = \begin{cases} 
  y + \frac{1}{2} & 0 < y \leq 1 \\
  0 & \text{elsewhere} 
\end{cases} \).
Same Ex: CDF from the PDF

- We just showed \( f_{X,Y}(x, y) = \begin{cases} x + y & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)

- In general, the CDF of \( X, Y \) is

\[
F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} \, d\hat{y}
\]

- If \( x \geq 1 \) and \( y \geq 1 \), then \( F_{X,Y}(x, y) = 1 \).

- If \( x < 0 \) or \( y < 0 \), then \( F_{X,Y}(x, y) = 0 \).
**Ex: CDF from the PDF (continued)**

- \( f_{X,Y}(x, y) = \begin{cases} x + y & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)

- In general, the CDF of \( X, Y \) is

  \[
  F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} \, d\hat{y}
  \]

- If \( 0 < x, y \leq 1 \), then

  \[
  F_{X,Y}(x, y) = \int_{0}^{y} \int_{0}^{x} (\hat{x} + \hat{y}) \, d\hat{x} \, d\hat{y} = \int_{0}^{y} \left( \frac{x^2}{2} + \hat{y}x \right) \, d\hat{y}
  \]

  \[
  = \frac{x^2 y}{2} + \frac{xy^2}{2} = \frac{xy}{2} (x + y)
  \]
Ex: CDF from the PDF continued

- \( f_{X,Y}(x, y) = \begin{cases} x + y & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)

- In general, the CDF of \( X, Y \) is

\[
F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\hat{x}, \hat{y}) \, d\hat{x} \, d\hat{y}
\]

- If \( 0 < x \leq 1 \) and \( y \geq 1 \), then

\[
F_{X,Y}(x, y) = \int_{0}^{1} \int_{0}^{x} (\hat{x} + \hat{y}) \, d\hat{x} \, d\hat{y} = \int_{0}^{1} \left( \frac{x^2}{2} + \hat{y}x \right) d\hat{y}
\]

\[
= \frac{x^2}{2} + \frac{x}{2} = \frac{x}{2}(x + 1)
\]
Ex: CDF from the PDF

Summary

\[ f_{X,Y}(x, y) = \begin{cases} 
  x + y & 0 < x, y \leq 1 \\
  0 & \text{elsewhere}
\end{cases} \]

\[ F_{X,Y}(x, y) = P[X \leq x, Y \leq y] = \begin{cases} 
  0 & \text{for } x \leq 0 \text{ or } y \leq 0 \\
  \frac{1}{2} xy(x + y) & \text{for } 0 < x, y \leq 1 \\
  \frac{1}{2} x(x + 1) & \text{for } 0 \leq x \leq 1, y > 1 \\
  \frac{1}{2} y(1 + y) & \text{for } 0 \leq y \leq 1, x > 1 \\
  1 & \text{for } x > 1 \text{ and } y > 1
\end{cases} \]
Independence of Two RVs

Section 5.5
Independence of $X$ and $Y$

- **Def**: RV’s $X$ and $Y$ are **independent RVs** if for any subsets $A$, $B$ of $\mathbb{R}$ we have independence back in $S$:

  \[(1) \quad P[\{X \in A, Y \in B\}] = P[\{X \in A\}]P[\{Y \in B\}]\]

- It can be shown that (1) is equivalent to

  \[F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \text{for all } x, y \in \mathbb{R}.\]
  
  (the joint CDF = product of marginal CDFs)

- It can also be shown that (1) is equivalent to

  \[f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \text{for all } x, y \in \mathbb{R}.\]
  
  (the joint PDF = product of marginal PDFs)
Independence of $X$ and $Y$ continued

- $X$ and $Y$ are independent RVs if for any sets $A$ and $B$

\[(1) \quad P[\{X \in A, Y \in B\}] = P[\{X \in A\}]P[\{Y \in B\}]\]

- If $X$ and $Y$ are discrete RVs, then (1) is also equivalent to

\[p_{X,Y}(x_j, y_k) = P[X = x_j, Y = y_k]\]

\[= P[X = x_j]P[Y = y_k]\]

\[= p_X(x_j)p_Y(y_k)\]

where $S_{XY} = S_X \times S_Y = \{(x_j, y_k) | j = 1, 2, \ldots, k = 1, 2, \ldots\}$

(the joint PMF = product of marginal PMFs)
Recall: Ex 1: Two Discrete RVs

- Toss a red die and a blue die. Let $X = \# \text{ on red die}$. Let $Y = \# \text{ on blue die}$. $S_X = S_Y = \{1, 2, 3, 4, 5, 6\}$

- If we assume the dice are fair, then we have
  
  $$p_{X,Y}(j,k) = P[X = j, Y = k] = \frac{1}{36} \quad \text{for } j,k = 1,2,3,4,5,6.$$ 

- We also have
  
  $$p_X(j) p_Y(k) = P[X = j] P[Y = k]$$
  
  $$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = p_{X,Y}(j,k)$$
  
  for $j,k = 1,2,3,4,5,6$.

- $X$ and $Y$ are independent!
Recall: Ex 2: Two Discrete RVs

- Toss two identical fair dice. Let \( X = \) the larger of the 2 #s. Let \( Y = \) the smaller.

- What was (1,1) remains (1,1), but (1,2) and (2,1) now are both (2,1), so \( S_{XY} = \{(j,k)|1 \leq k \leq j \leq 6\}\)

- And

\[
p_{X,Y}(j,k) = \begin{cases} \frac{1}{36} & j = k = 1, 2, 3, 4, 5, 6 \\ \frac{1}{18} & 1 \leq k < j \leq 6 \end{cases}
\]

- \( X \) and \( Y \) are not independent since we have for \( j \neq k \)

\[
p_{X,Y}(j,k) \neq p_X(j)p_Y(k)
\]
Recall: Uniform on Unit Square

- Here we want the prob of a set to be equal to its area.

\[
F_{XY}(x, y) = P[X \leq x, Y \leq y] = \begin{cases} 
0 & \text{for } x < 0 \text{ or } y < 0 \\
xy & \text{for } 0 \leq x, y \leq 1 \\
x & \text{for } 0 \leq x \leq 1, \ y > 1 \\
y & \text{for } 0 \leq y \leq 1, \ x > 1 \\
1 & \text{for } x > 1 \text{ and } y > 1
\end{cases}
\]
PDF of Uniform

- We showed that the joint PDF is

\[
f_{X,Y}(x, y) = \begin{cases} 1 & 0 \leq x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases}
\]

- And the marginal PDFs are

\[
f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}
\]

\[
f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}
\]
Thus we have
\[ f_{X,Y}(x, y) = 1 = f_X(x) f_Y(y) \quad \text{for all } (x, y) \in [0,1] \times [0,1]. \]

and
\[ f_{X,Y}(x, y) = 0 = f_X(x) f_Y(y) \quad \text{for all } (x, y) \notin [0,1] \times [0,1]. \]

So \( f_{X,Y}(x, y) = f_X(x) f_Y(y) \) for all \( x, y \in \mathbb{R} \).

and \( X \) and \( Y \) are independent.

Can also show here:
\[ F_{X,Y}(x, y) = F_X(x) F_Y(y) \quad \text{for all } x, y \in \mathbb{R}. \]
Ex: Two Exponential RVs

- Recall: \( F_{XY}(x, y) = \begin{cases} (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y}) & \text{for } x, y > 0 \\ 0 & \text{elsewhere} \end{cases} \)

- Here we can see directly that

\[
F_{X,Y}(x, y) = F_X(x) F_Y(y) \quad \text{for all } x, y \in \mathbb{R}.
\]

- So \( X \) and \( Y \) are independent.

- Also for \( x > 0 \) and \( y > 0 \),

\[
f_{X,Y}(x, y) = (\lambda_1 e^{-\lambda_1 x})(\lambda_2 e^{-\lambda_2 y}) = f_X(x) f_Y(y)
\]
Recall: Two Jointly Gaussian RVs

- RVs $X$ and $Y$ are \textbf{jointly Gaussian} if $f_{XY}(x,y) =$

\[
\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp \left\{ \left( \frac{-1}{2(1-\rho^2)} \right) \left[ \frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right] \right\}
\]

- The marginal distribution of $X$ is Gaussian with mean $\mu_1$ and variance $\sigma_1^2$.

- The marginal distribution of $Y$ is Gaussian with mean $\mu_2$ and variance $\sigma_2^2$.

- $\rho$ is the correlation coefficient which we will discuss next time.

\[
\rho = \frac{\mathbb{E}[(X-\mu_1)(Y-\mu_2)]}{\sigma_1\sigma_2}
\]
Independence of Jointly Gaussian RVs

- If $\rho = 0$, then

$$f_{X,Y}(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2} \exp \left\{ -\left[ \frac{(x - \mu_1)^2}{2\sigma_1^2} + \frac{(y - \mu_2)^2}{2\sigma_2^2} \right] \right\}$$

$$= \left( \frac{1}{\sqrt{2\pi \sigma_1}} \exp \left[ -\frac{(x - \mu_1)^2}{2\sigma_1^2} \right] \right) \left( \frac{1}{\sqrt{2\pi \sigma_2}} \exp \left[ -\frac{(y - \mu_2)^2}{2\sigma_2^2} \right] \right)$$

- So $f_{X,Y}(x, y) = f_X(x) f_Y(y)$.
- So $\rho = 0$ for jointly Gaussian implies independence!
- More next time on $\rho$. In particular, we will show $\rho \neq 0$ for jointly Gaussian implies not independent!
Another Example of a 2-RV PDF

- From earlier: \( f_{X,Y}(x, y) = \begin{cases} x + y & 0 < x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases} \)

- The marginal PDFs are
  \[
  f_X(x) = \begin{cases} x + \frac{1}{2} & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}
  \]
  \[
  f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 < y \leq 1 \\ 0 & \text{elsewhere} \end{cases}
  \]

- Take \( x = y = \frac{1}{4} \)
  \[
  f_{X,Y}(\frac{1}{4}, \frac{1}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq f_X(\frac{1}{4}) f_Y(\frac{1}{4}) = \left(\frac{1}{4} + \frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}
  \]

- So this \( X \) and \( Y \) are **not** independent!